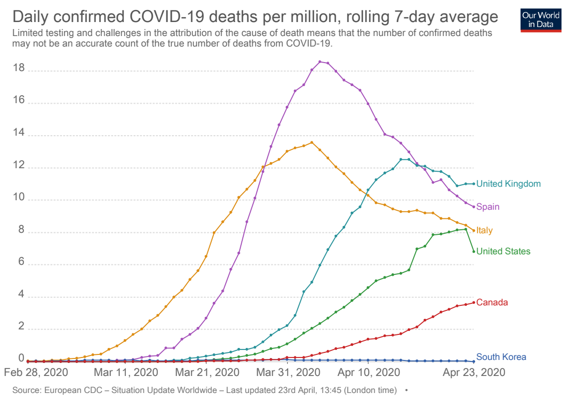
Extreme value statistics and the theory of rare events

Francesco Mori

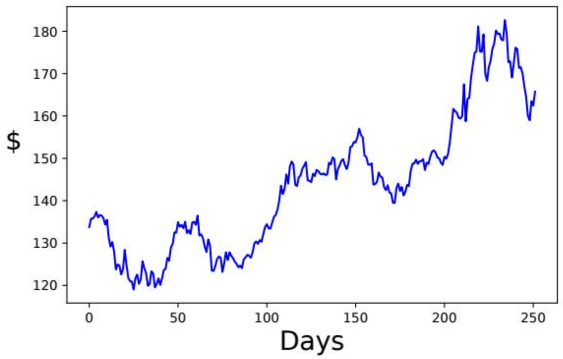
Saturday Mornings of Theoretical Physics
25/02/2023

Extreme value statistics: motivation

Applications to epidemics[1], finance [2], climate science [3]



From Our World in Data

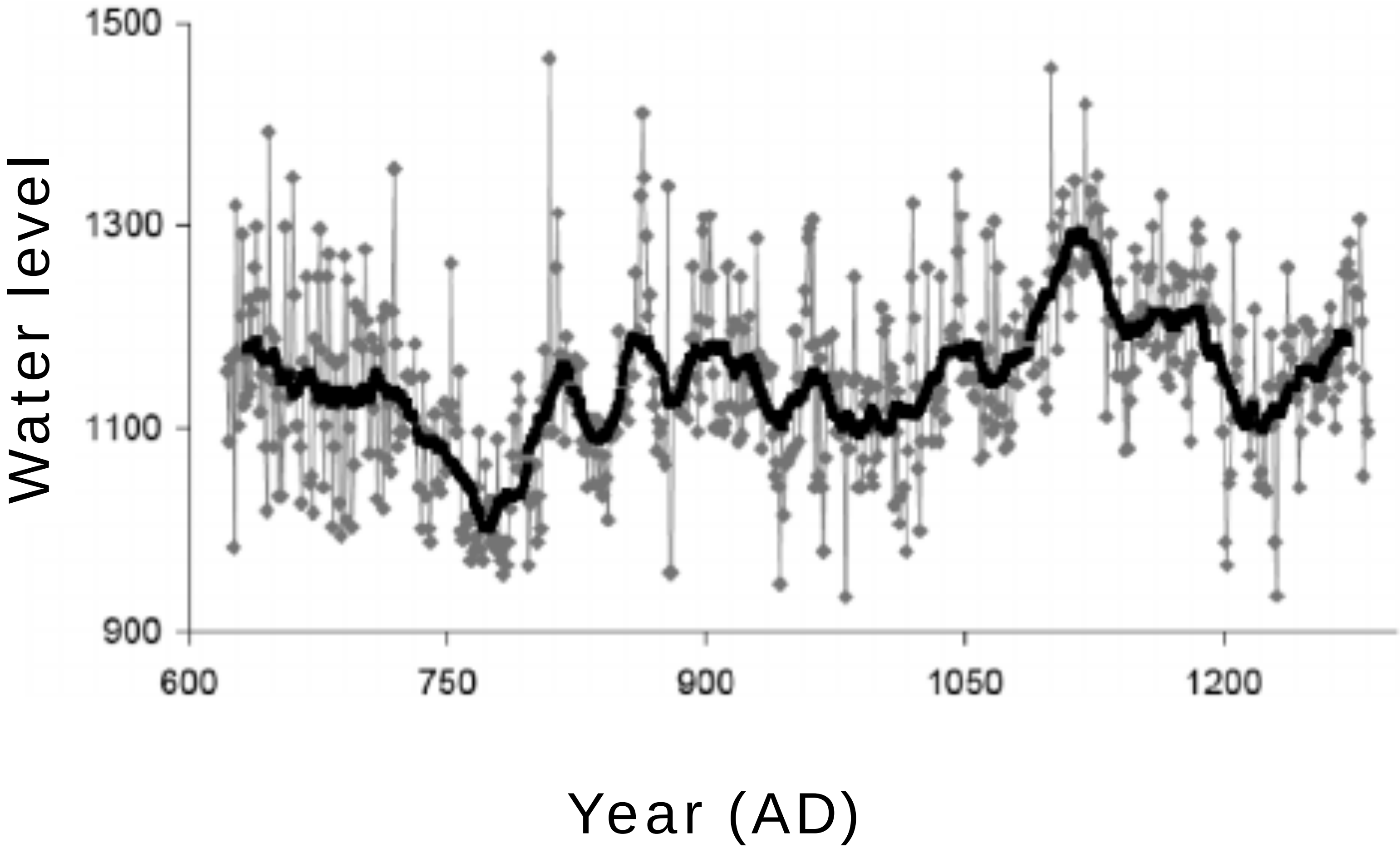


Apple price Jan 2021-Jan 2022 (from Yahoo finance)

From Wikipedia

[1] M. Marani, Marco, et al., *Proceedings of the National Academy of Sciences* **118**: e2105482118 (2021).
[2] S.N. Majumdar and J.-P. Bouchaud, Quantitative Finance 8, 753 (2008).
[3] G. Wergen and J. Krug, Europhys. Lett. 92, 30008 (2010).

Extreme value statistics: an example



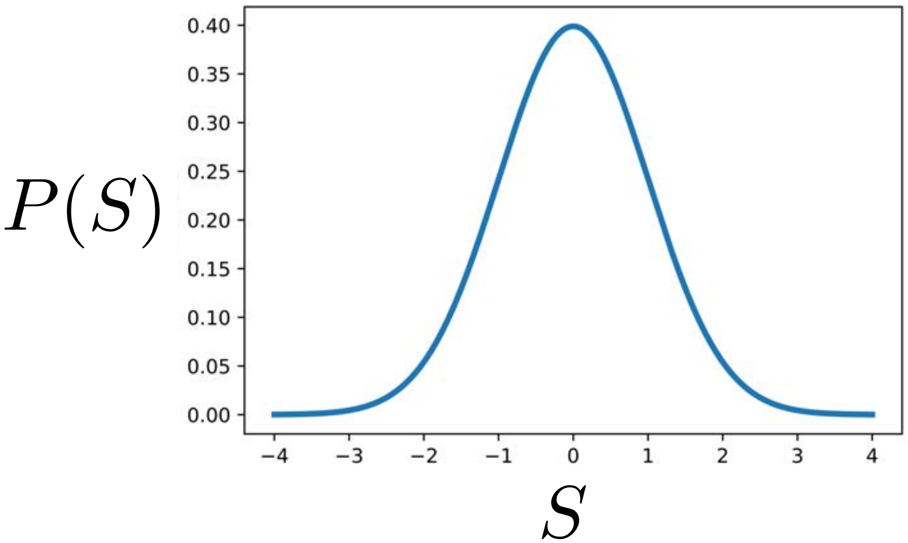
Di Baldassarre, Giuliano, et al., *Hydrological Sciences Journal,* 56.2 (2011): 199-211.

Central Limit Theorem

$$S_N = X_1 + X_2 + \ldots + X_N$$

$$P(S) = \text{Prob.}(S_N = S) \approx \frac{1}{\sqrt{2\pi\sigma^2 N}} \exp\left(-\frac{S^2}{2\sigma^2 N}\right)$$

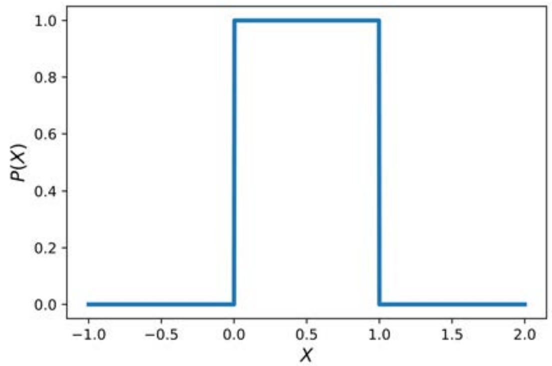
UNIVERSAL!



Gaussian distribution
(a.k.a. Normal distribution)

Central Limit Theorem
Numerical example

$$p(X) = \begin{cases} 1 & \text{if } 0 < x < 1\,, \\ 0 & \text{otherwise} \end{cases}$$



Central Limit Theorem
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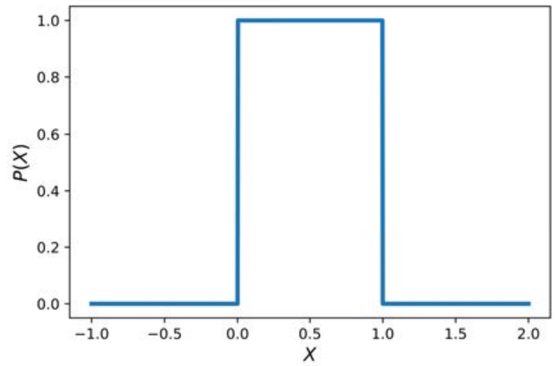
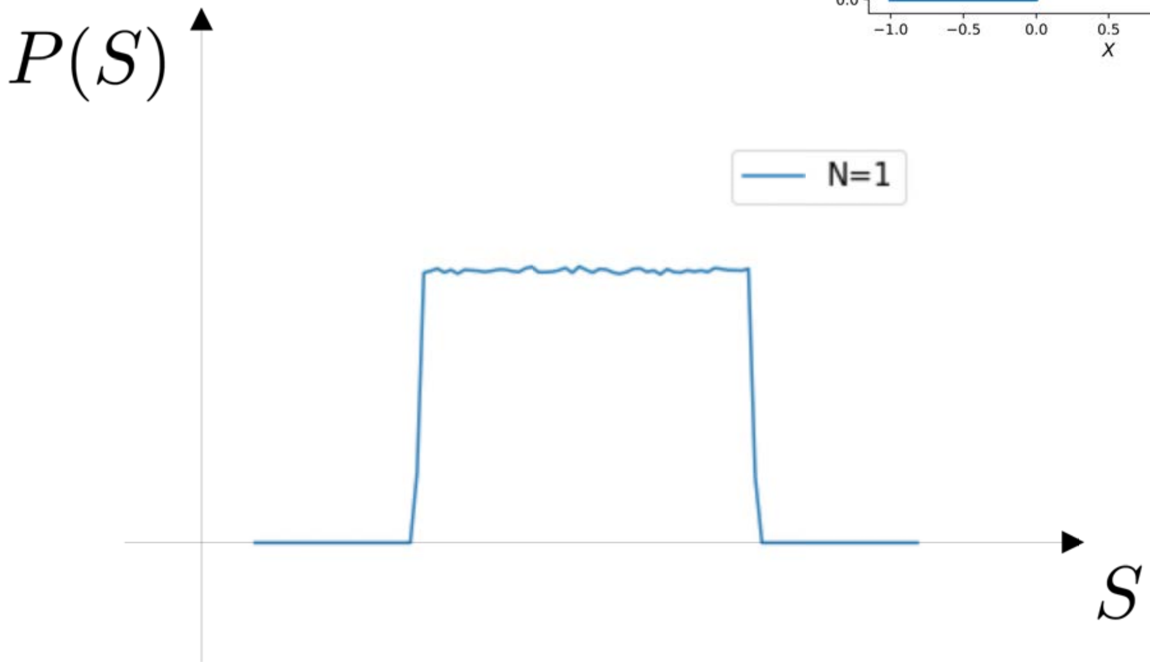
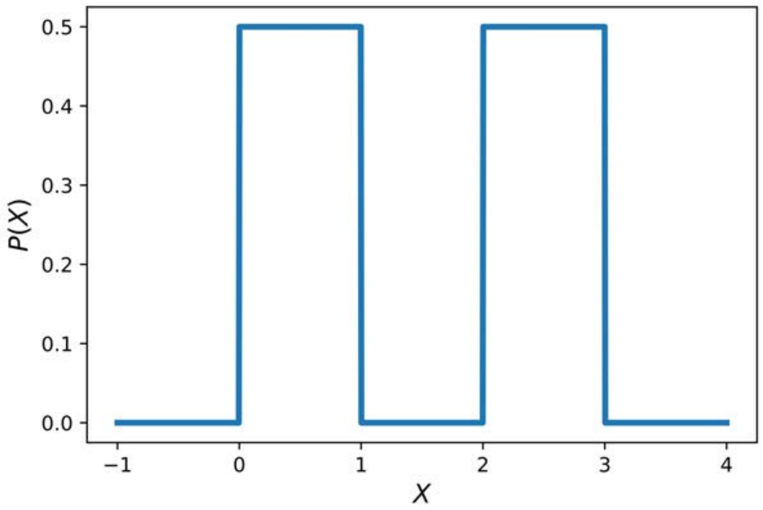




Central Limit Theorem
Numerical example

$$p(X) = \begin{cases} 1/2 & \text{if } 2 < x < 3\,, \\ 1/2 & \text{if } 0 < x < 1\,, \\ 0 & \text{otherwise} \end{cases}$$



Central Limit Theorem
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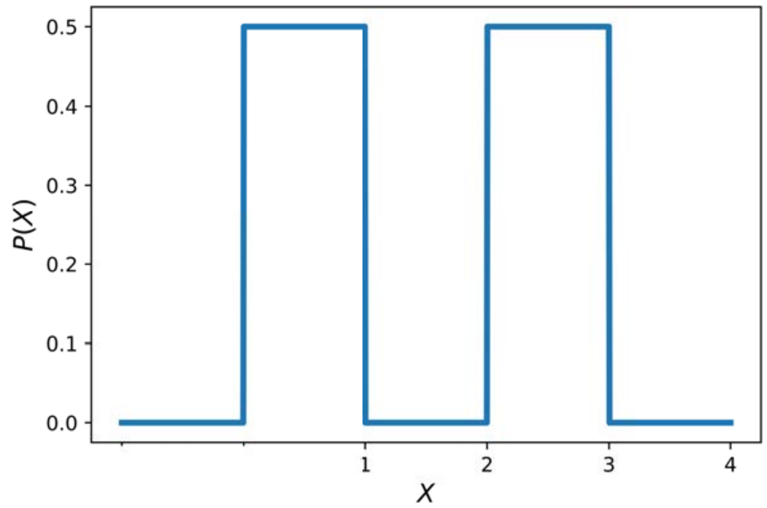
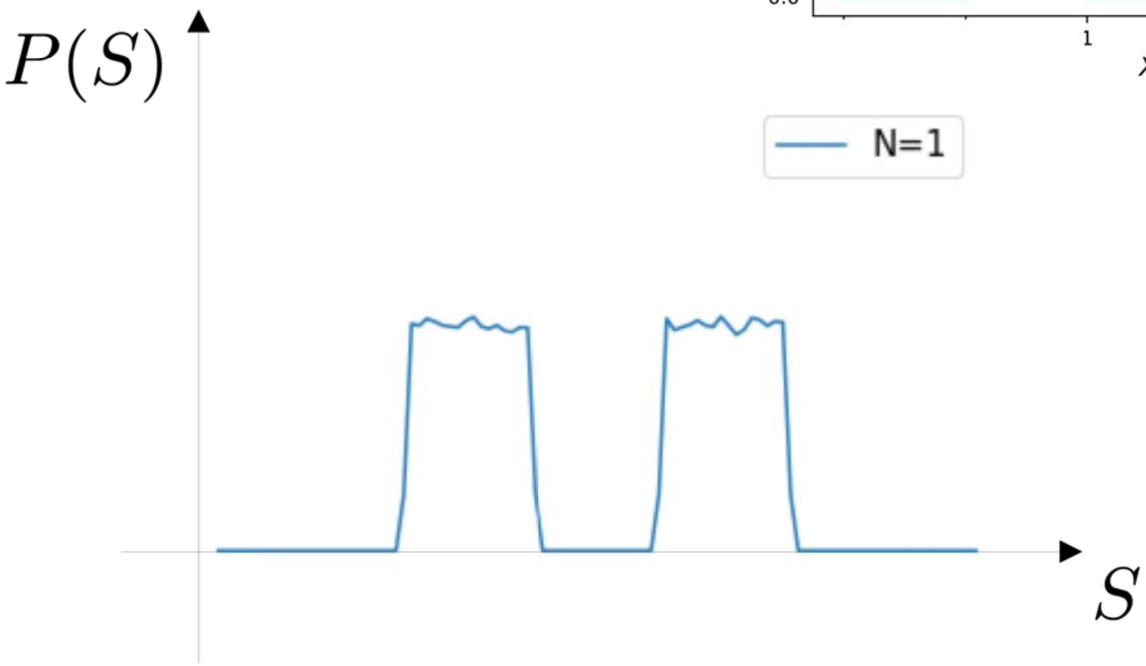
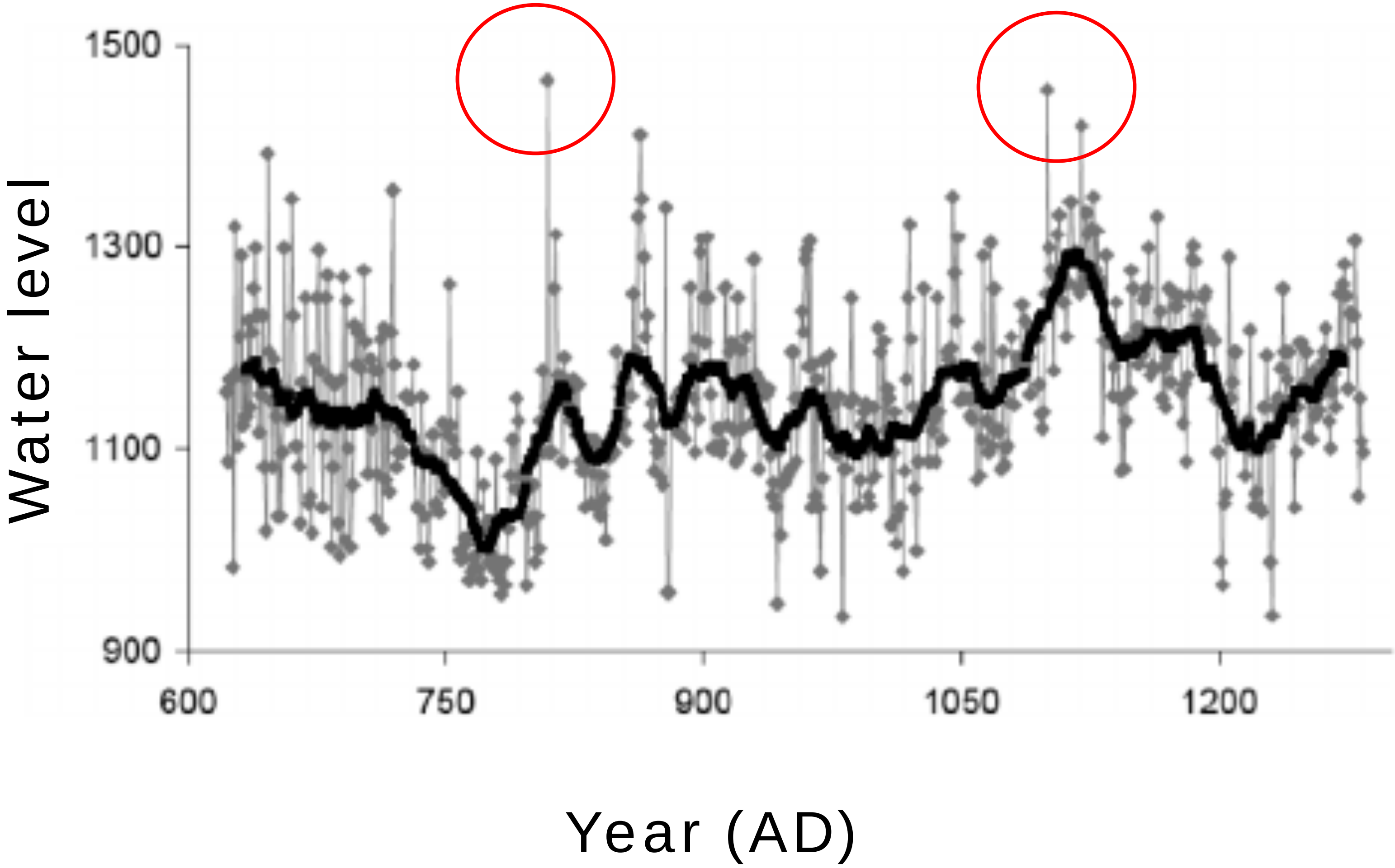


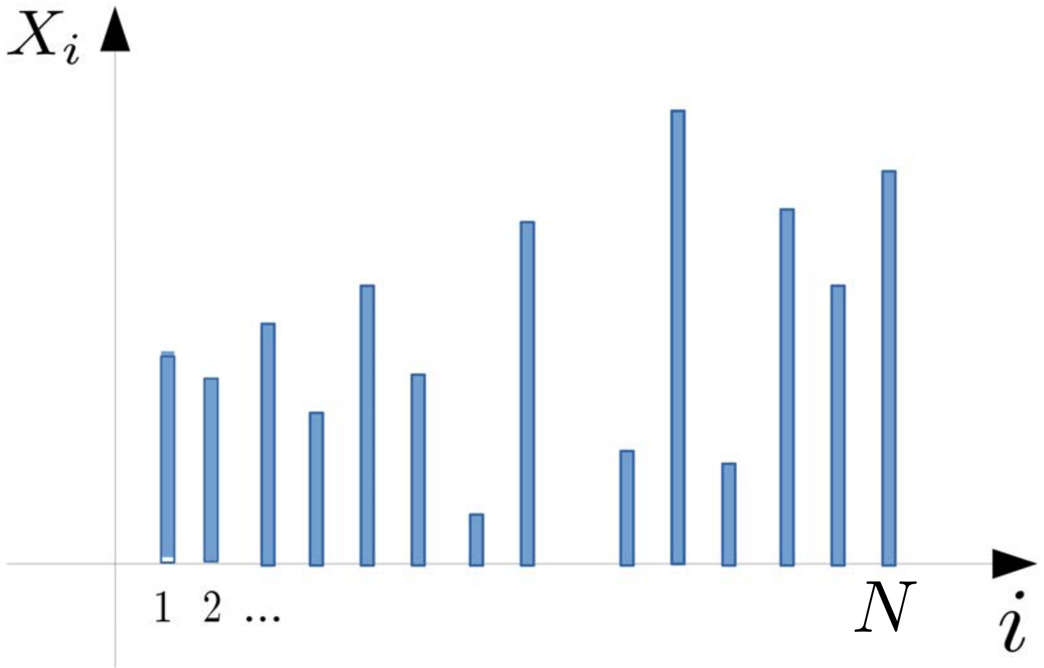


Extreme value statistics: an example



Di Baldassarre, Giuliano, et al., *Hydrological Sciences Journal,* 56.2 (2011): 199-211.

Extreme value statistics: setting



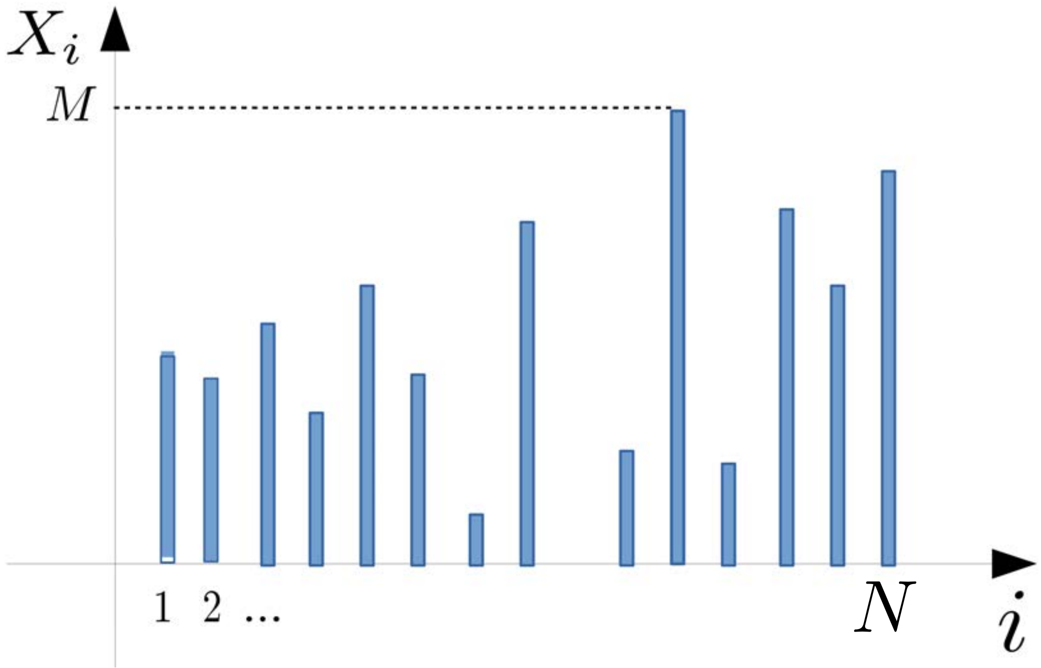
$$X_1\,, X_2\,, \ldots\,, X_N$$

random variables with distribution

$$P(X_1\,, \ldots\,, X_N)$$

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

Global Maximum

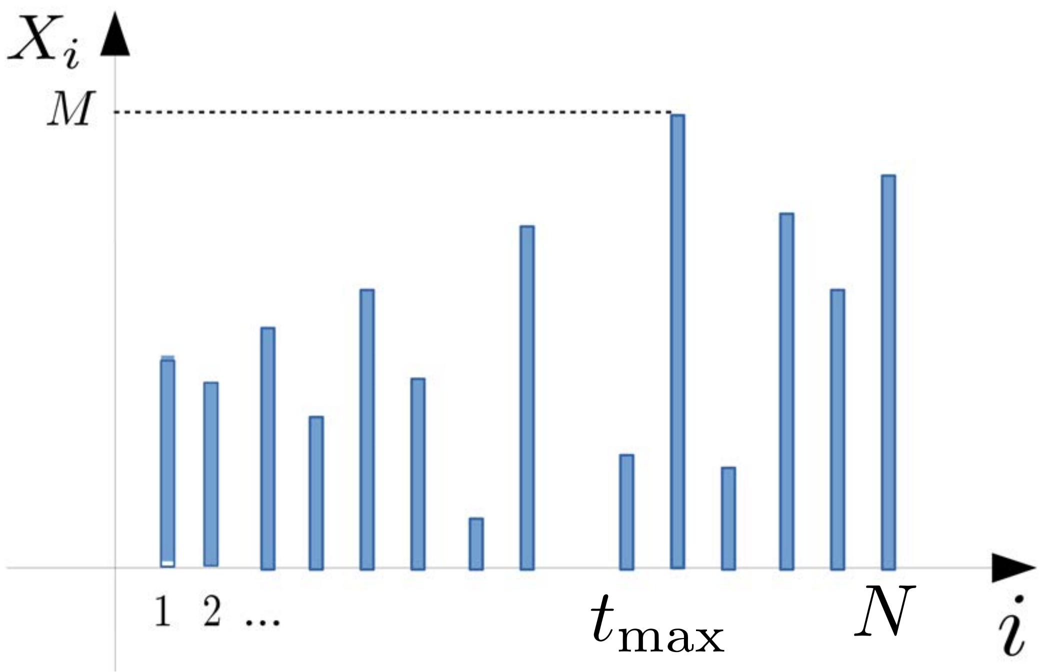


$$M = \max_{1 \le i \le N} X_i$$

Given $P(X_1, \ldots, X_N)$ what can I say about the distribution of *M*?

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

Time of the maximum



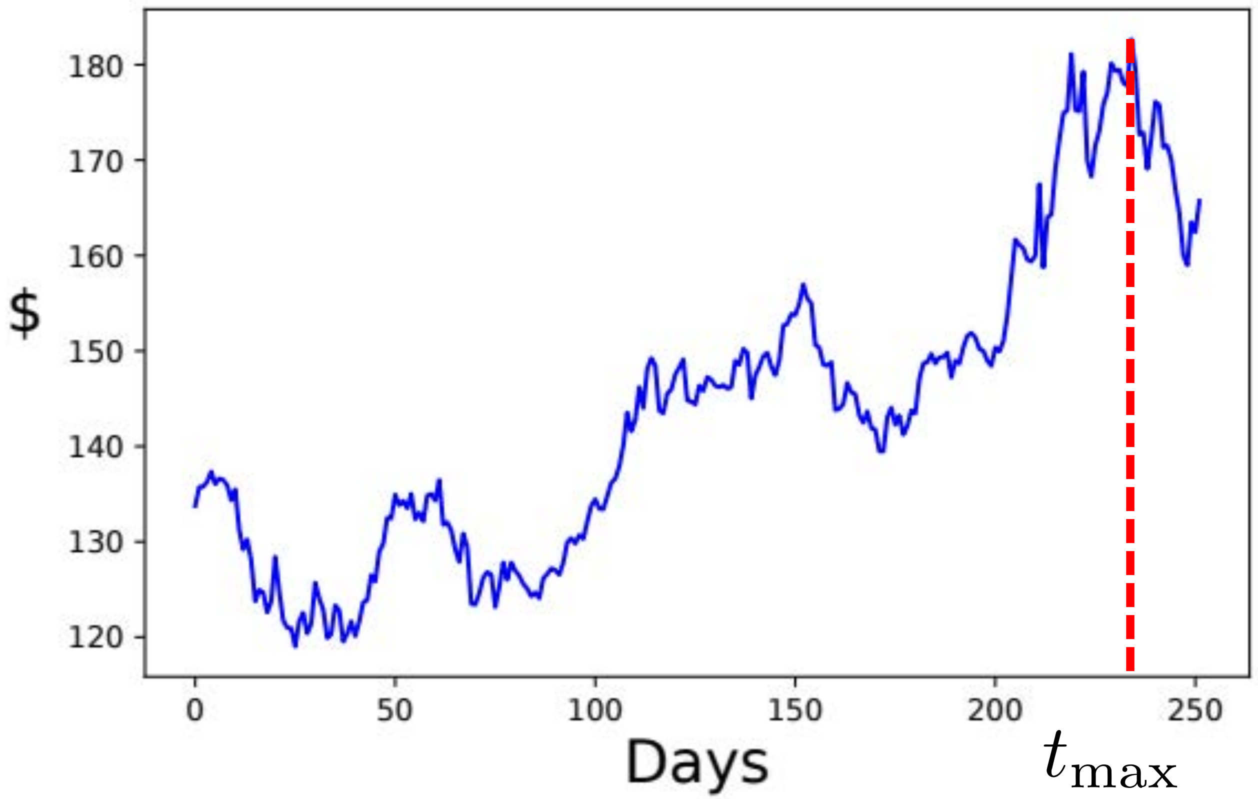
$$t_{\max} = \operatorname{argmax}_{1 \le i \le N} X_i$$

Given $P(X_1, \ldots, X_N)$ what can I say about the distribution of t_{max}?

Applications to finance, disordered systems, sports...

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

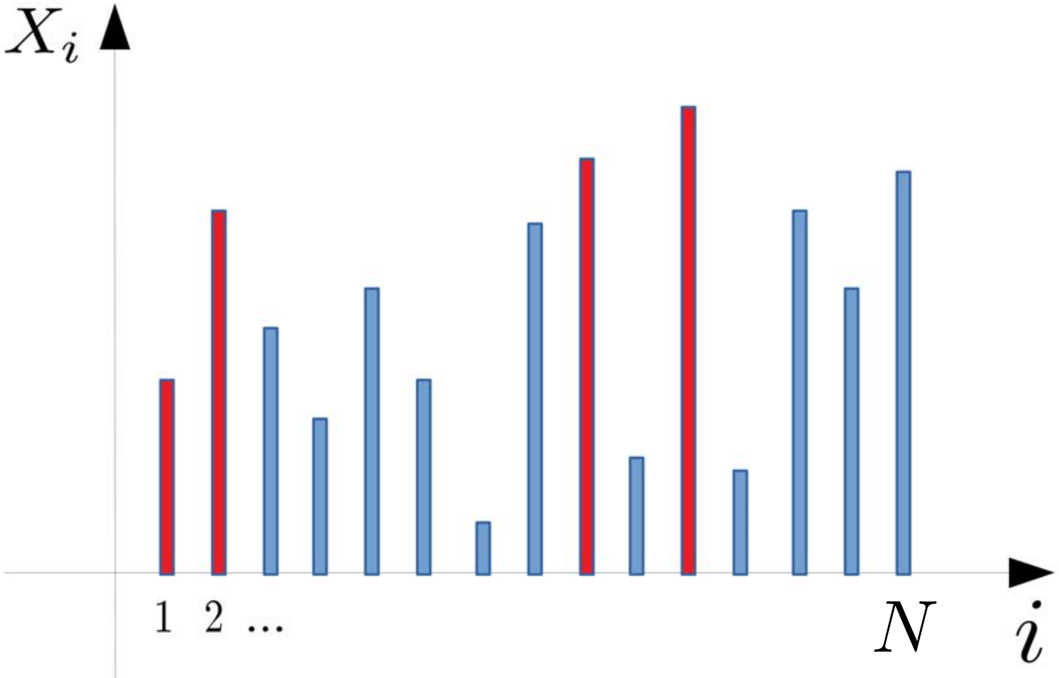
Time of the maximum

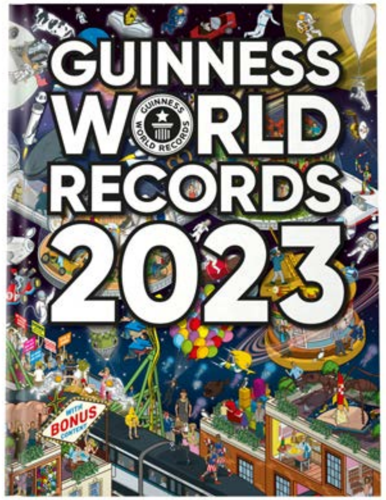


Apple price Jan 2021-Jan 2022
(from Yahoo finance)

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

Record statistics



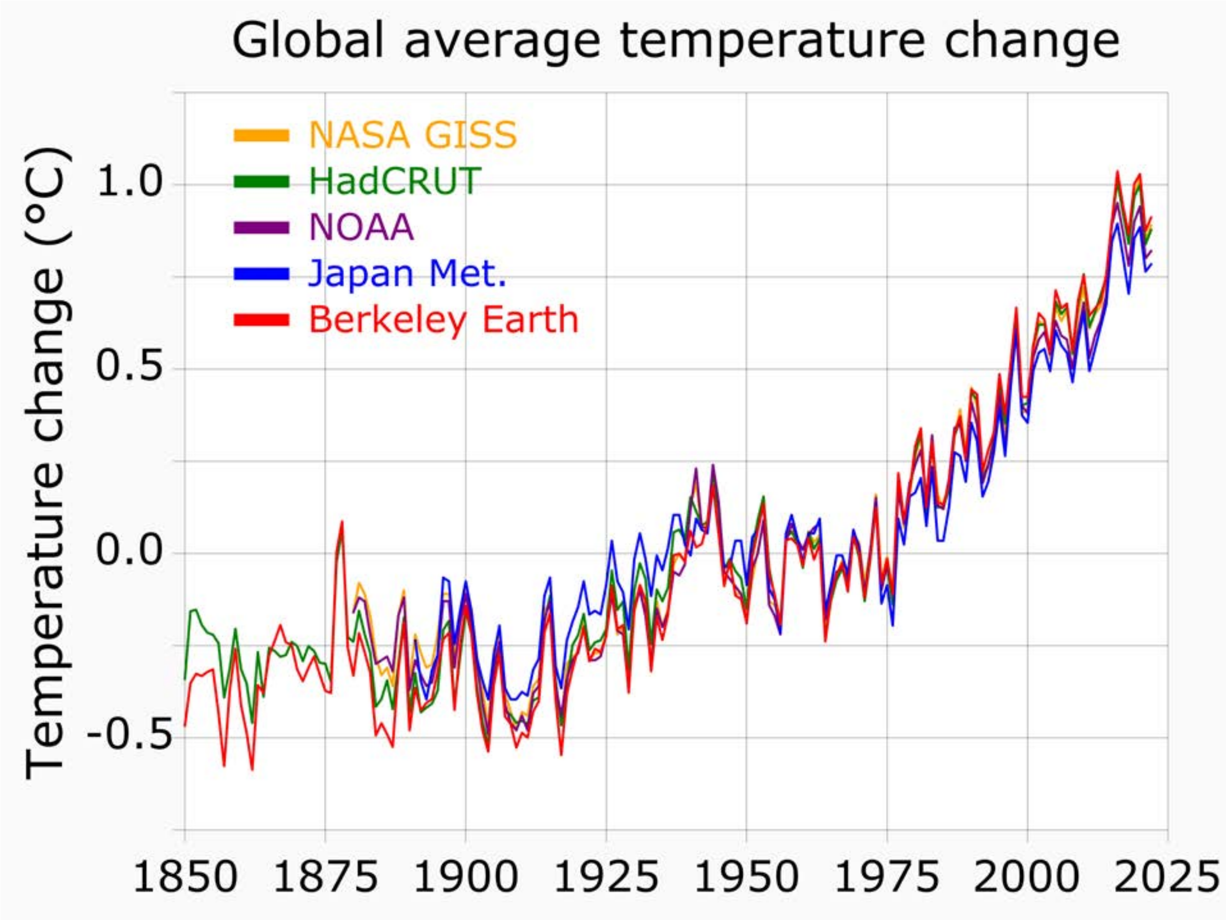


$$X_i \text{ is a record if } X_i > \{X_1\,,, X_2\,, \dots\,, X_{i-1}\}$$

Applications to climate science, sports, evolution, insurance policies,...

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

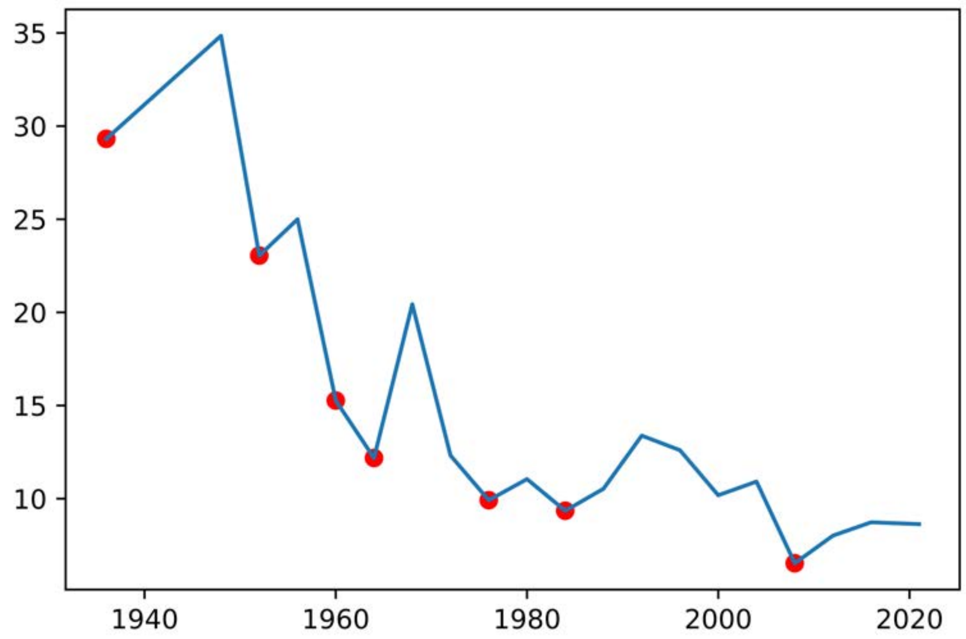
Record statistics in climate science



From Wikipedia

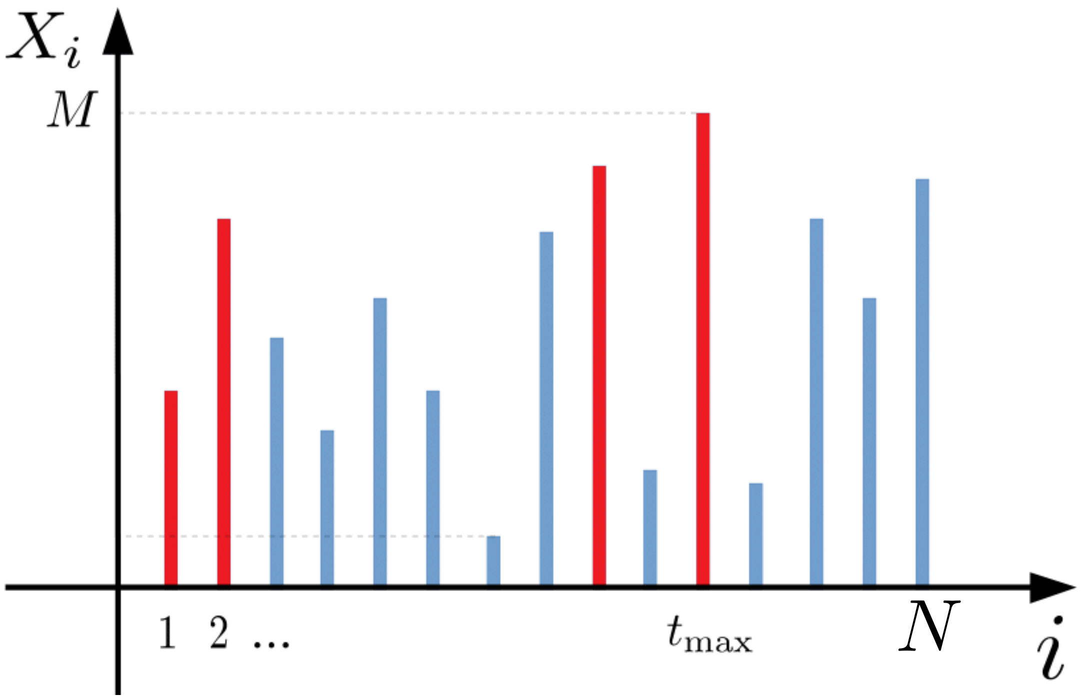
Marathon world record

Time above
2 hours [min]



Year

Extreme value statistics: setting



Global maximum

$$M = \max_{1 \le i \le N} X_i$$

Time of the maximum

$$t_{\max} = \mathrm{argmax}_{1 \le i \le N} X_i$$

Record statistics

N_R (Number of records)

[1] S. N. Majumdar, A. Pal, and G. Schehr, Physics Reports 840, 1 (2020).

Independent and identically distributed variables

$$P(X_1, \ldots, X_N) = \prod_{i=1}^{N} p(X_i)$$

Joint probability distribution

Marginal probability distribution

No correlations

Example: Derrida's random energy model [1]

[1] Derrida, Bernard. "Random-energy model: An exactly solvable model of disordered systems." Physical Review B 24.5 (1981): 2613.

Extreme Value Theorem

$$M = \max_{1 \le i \le N} X_i$$

Gumbel

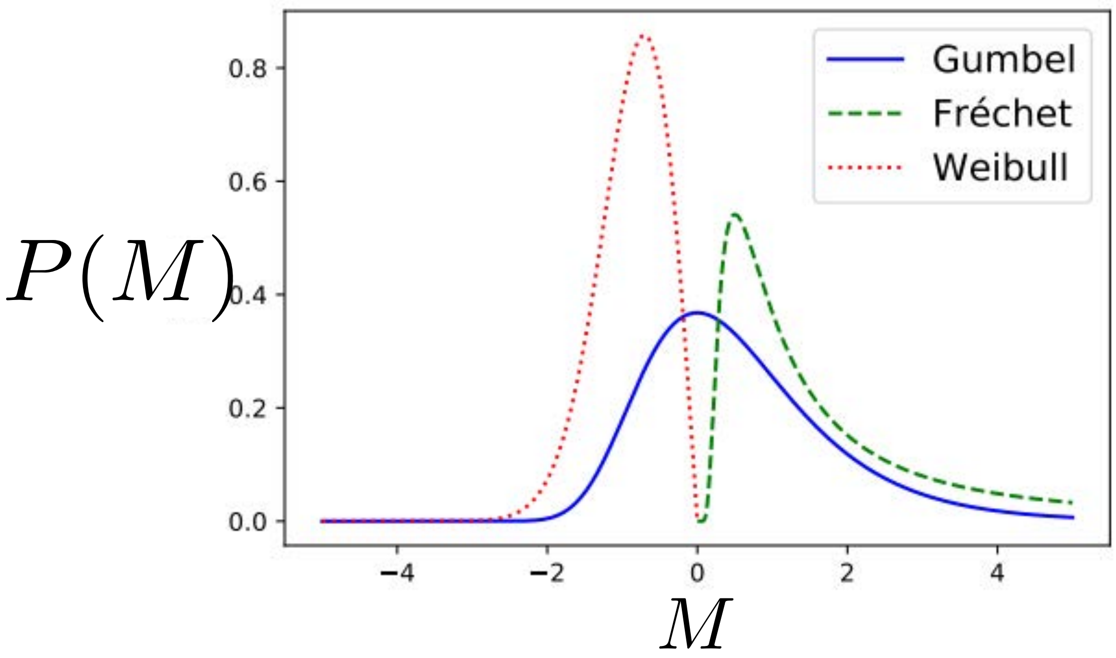
p(X) decays exponentially fast for large *x*

Fréchet

p(X) decays as a power law for large *x*

Weibull

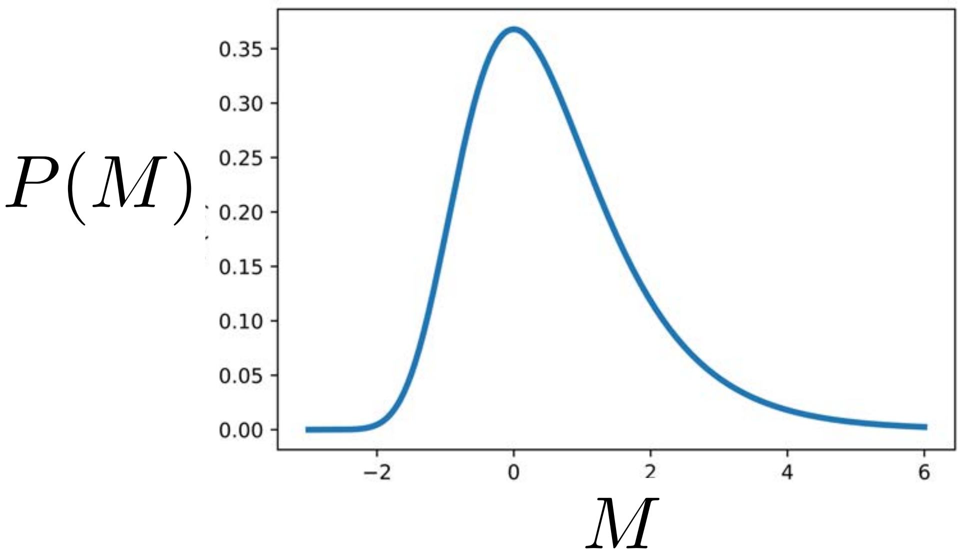
p(X) has an upper bounded support



Gumbel distribution

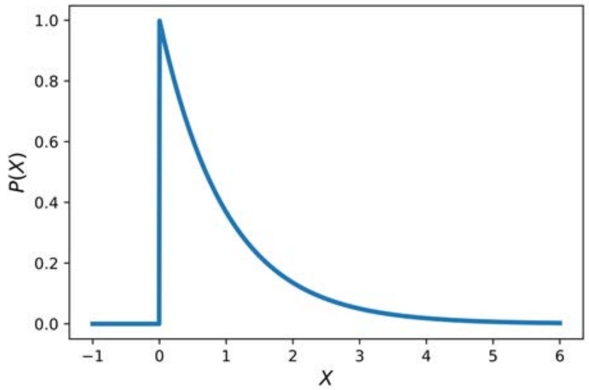
p(X) decays exponentially fast for large *x*

$$P(M) = e^{-M-e^{-M}}$$



Extreme value distribution
Numerical example

$$p(X) = \begin{cases} e^{-X} & \text{if } X > 0\,, \\ 0 & \text{otherwise} \end{cases}$$



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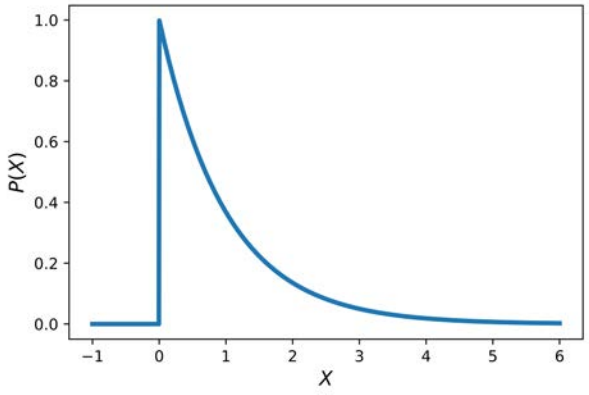
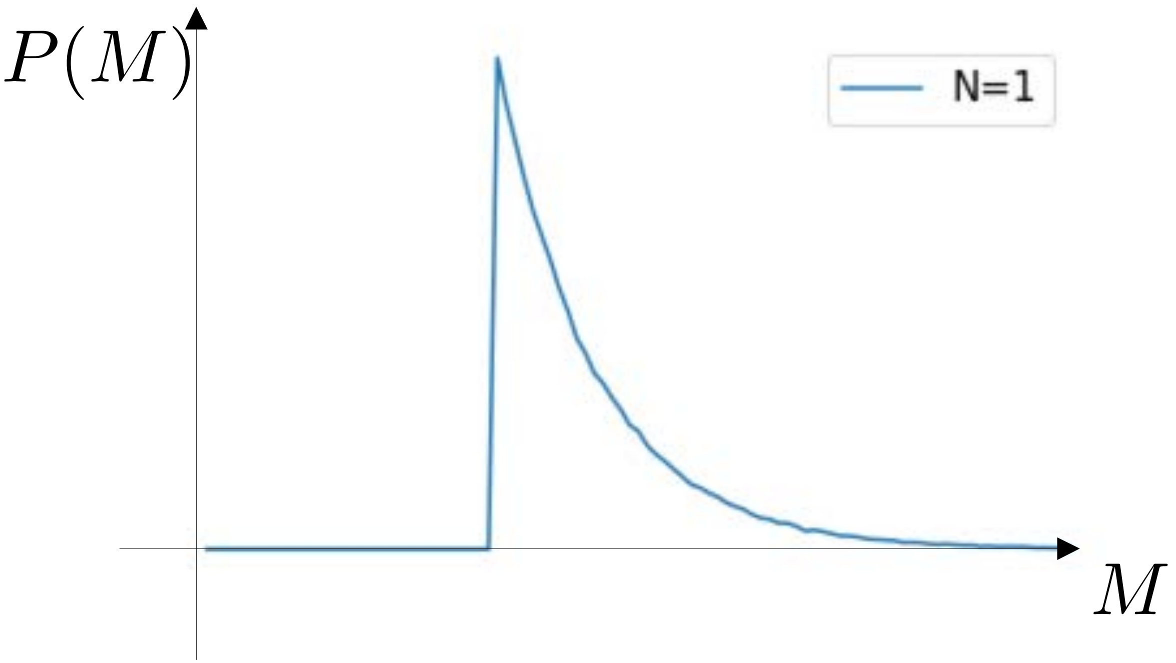
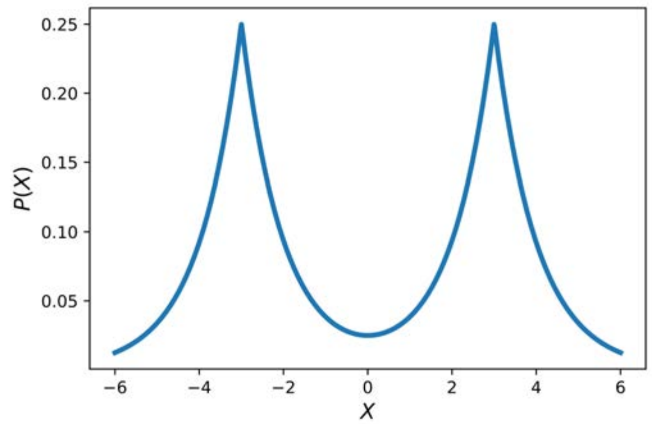




Extreme value distribution
Numerical example

$$p(X) = \frac{1}{4}\left(e^{-|X-3|} + e^{-|X+3|}\right)$$



Extreme value distribution
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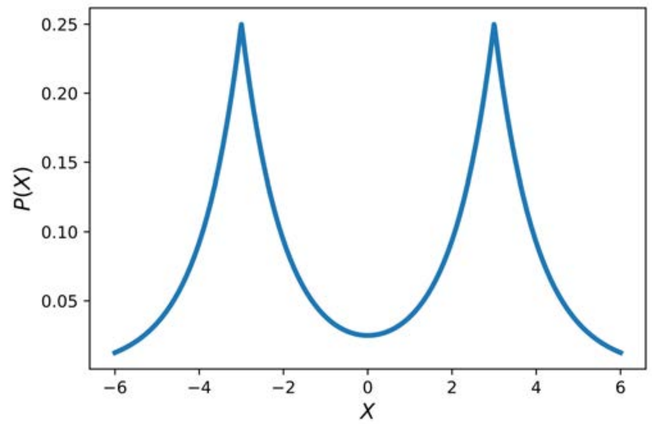
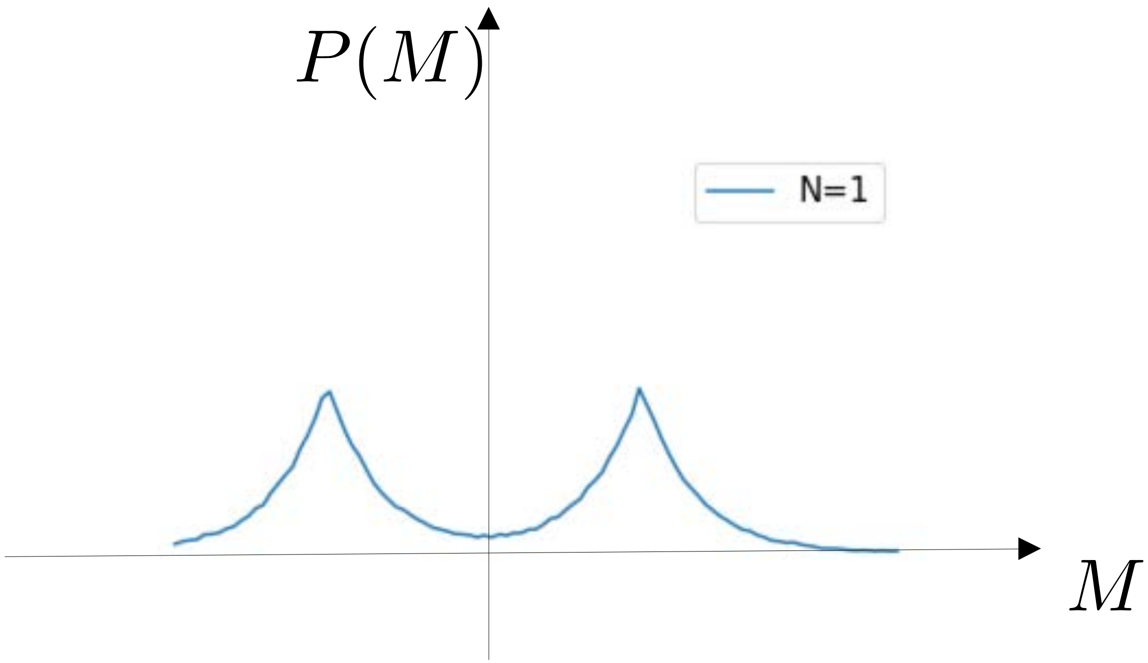




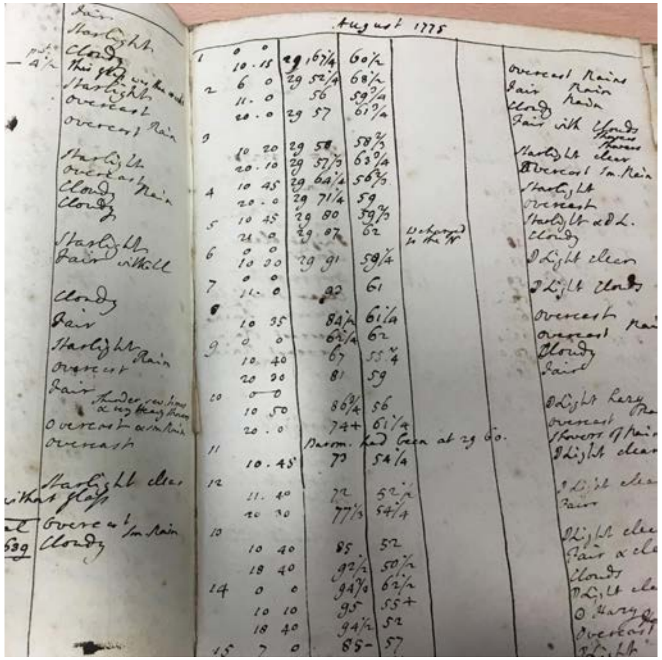
Does it work on real data?

Radcliffe Observatory

Images from Green Templeton College

Does it work on real data?

Radcliffe Meteorological Station



Images from Green Templeton College

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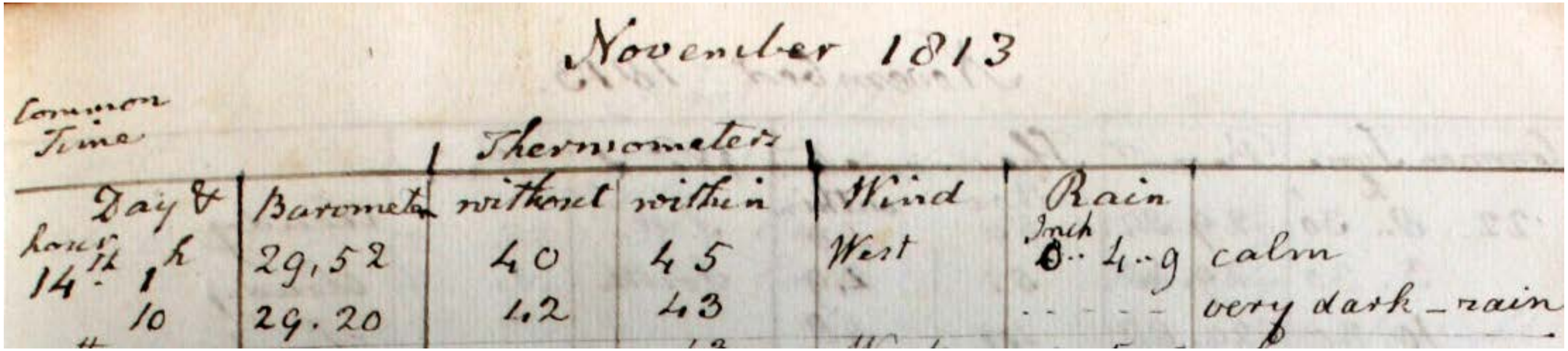


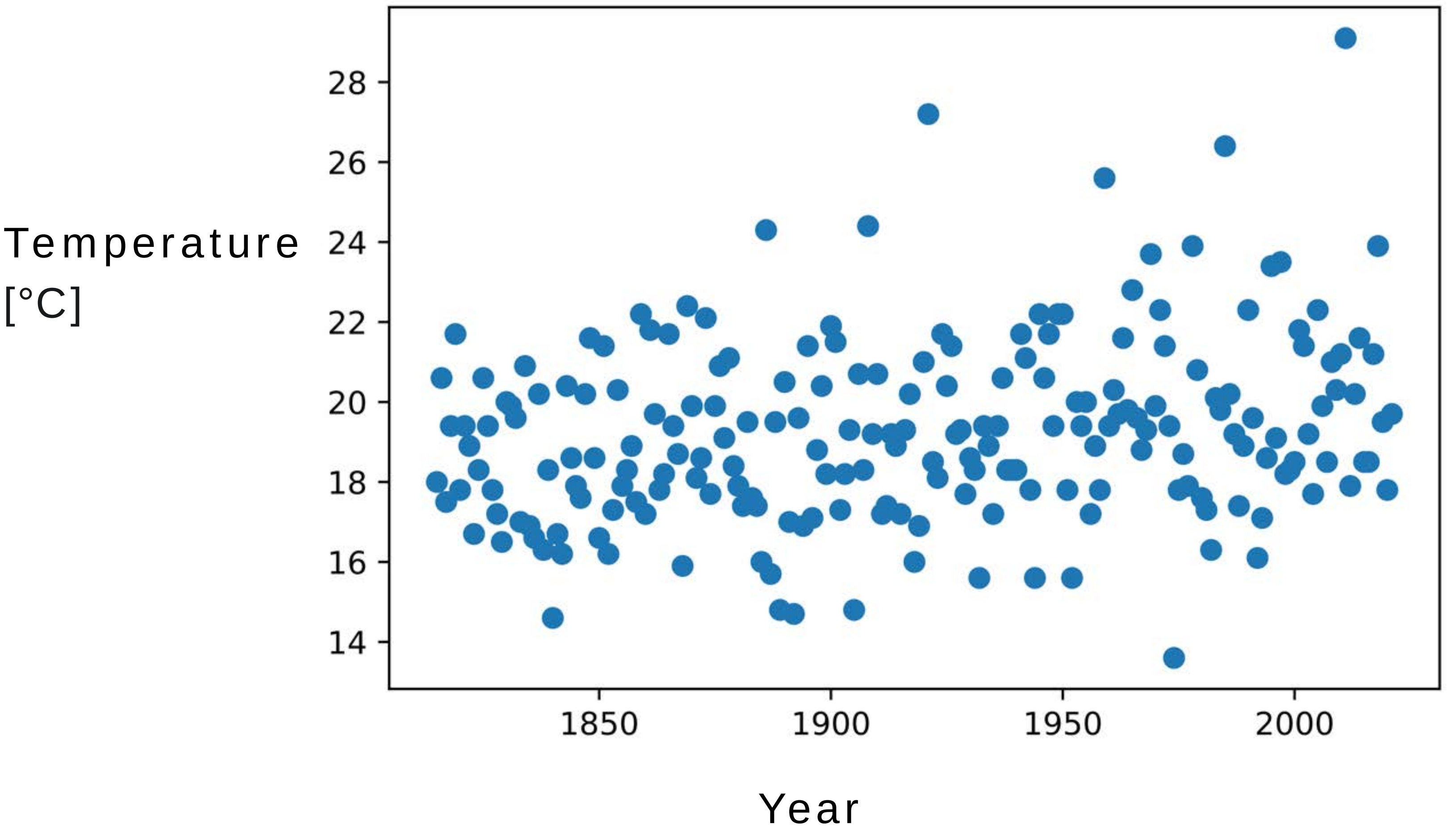
November 1813

Common Time Day & hours	Barometer	Thermometers without	Thermometers within	Wind	Rain Inch	
14th 1h	29,52	40	45	West	0..4..9	calm
10	29.20	42	43		- - - - -	very dark _ rain

YYYY	MM	DD	Tmax °C	Tmin °C	Daily Tmean °C	Daily range degC	Grass min °C	Air frost 0/1	Ground frost 0/1	Max ≥ 25.0°C	Max ≥ 30.0°C	Min ≥ 15.0 °C	Max < 0 °C
1815	1	1	6.6	-1.5	2.6	8.1		1		0	0	0	0
1815	1	2	4.9	-3.2	0.9	8.1		1		0	0	0	0
1815	1	3	2.6	-5.6	-1.5	8.2		1		0	0	0	0
1815	1	4	2.1	-6.1	-2	8.2		1		0	0	0	0
1815	1	5	1	-7.2	-3.1	8.2		1		0	0	0	0
1815	1	6	1.5	-6.6	-2.6	8.1		1		0	0	0	0
1815	1	7	-0.7	-9	-4.9	8.3		1		0	0	0	1
1815	1	8	4.9	-3.2	0.9	8.1		1		0	0	0	0
1815	1	9	1	-7.2	-3.1	8.2		1		0	0	0	0
1815	1	10	7.7	-0.4	3.7	8.1		1		0	0	0	0

Images from the School of Geography and Environment

Maximal temperature in October



Data from https://www.geog.ox.ac.uk/

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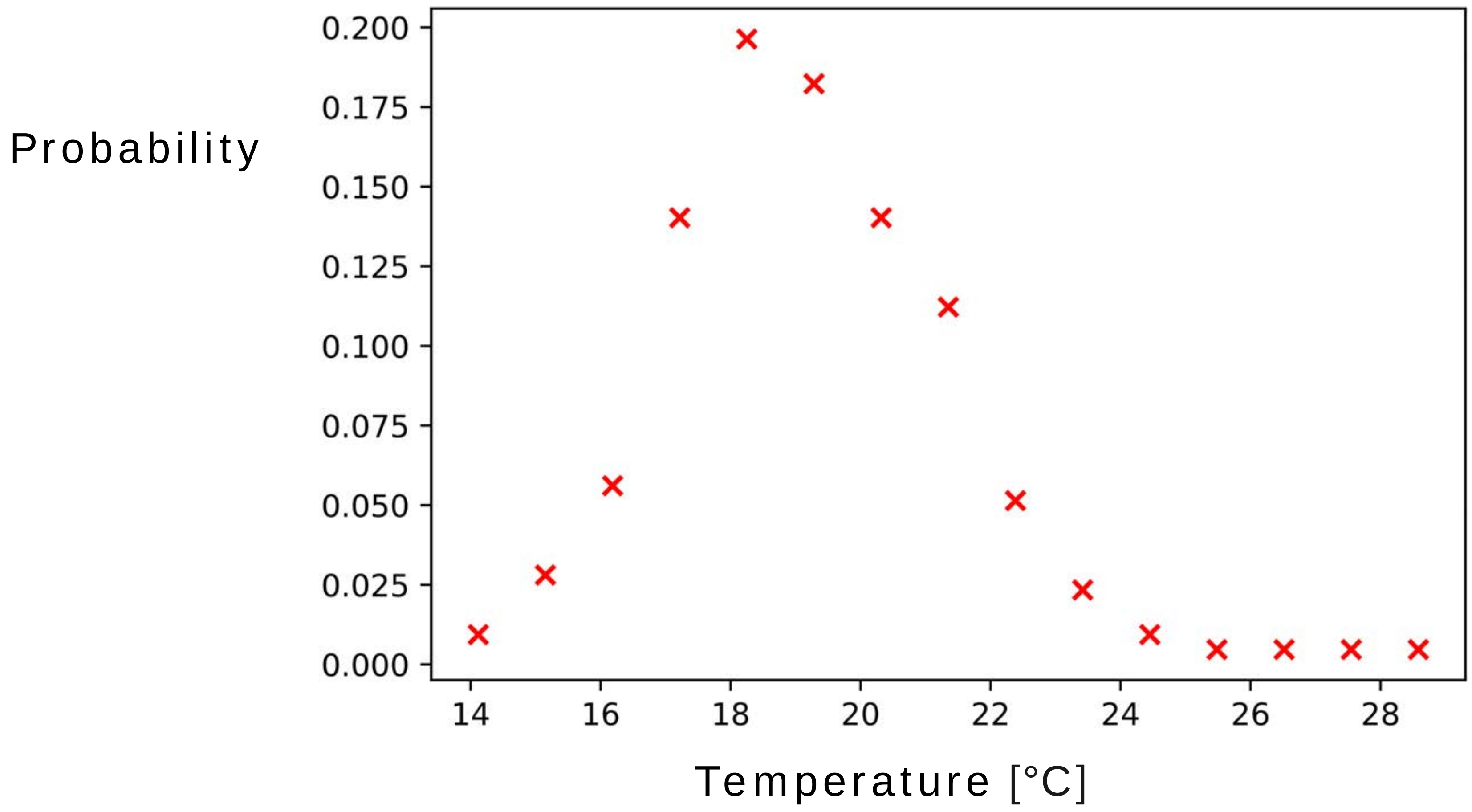

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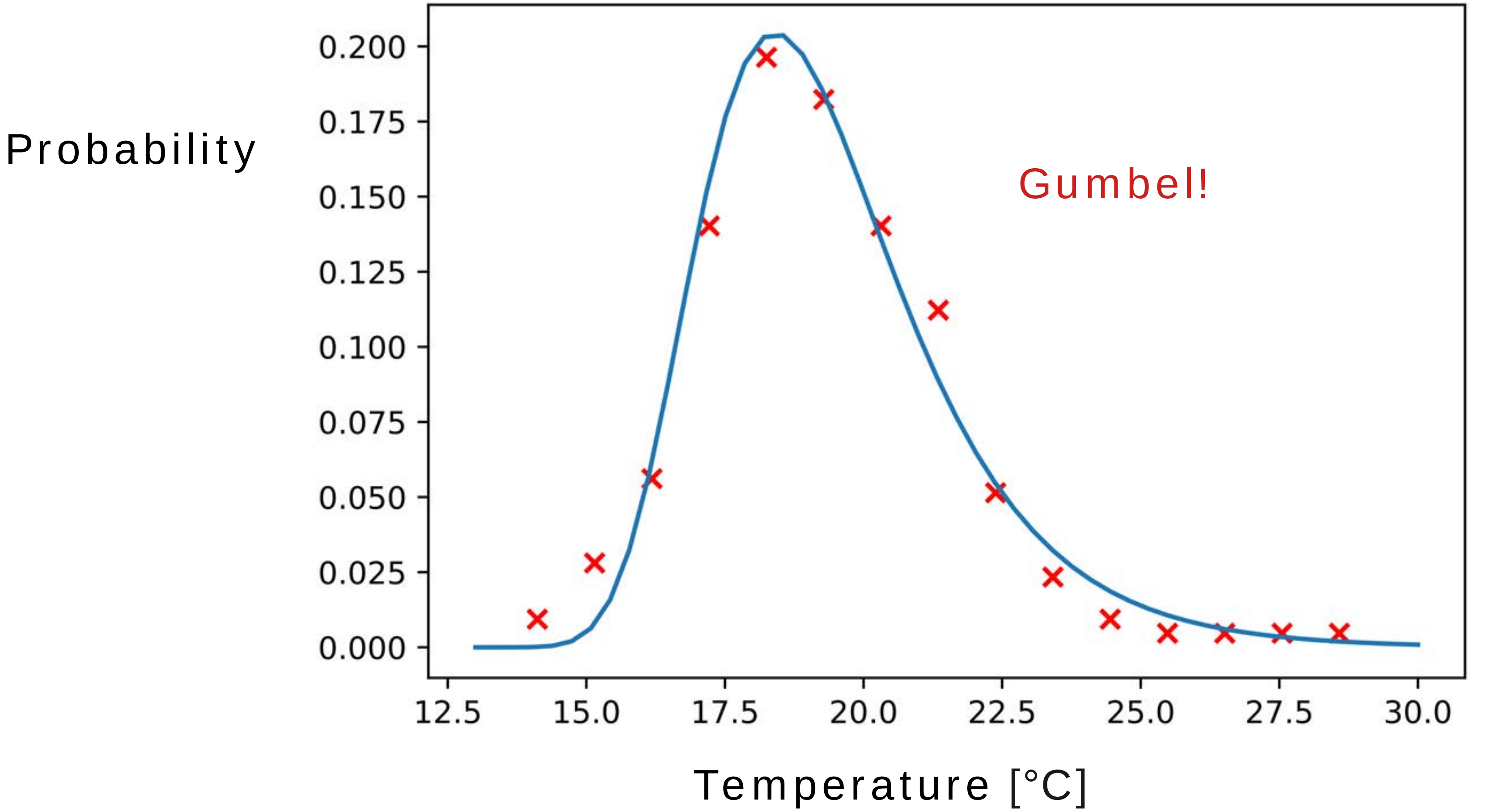

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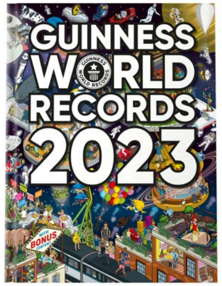
Independent variables: record statistics

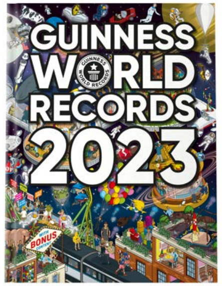


$$P(X_1, \ldots, X_N) = \prod_{i=1}^{N} p(X_i)$$

$$X_i \text{ is a record if } X_i > \{X_1, , X_2, \ldots, X_{i-1}\}$$

Given a sequence of *N* random numbers, how many records do we expect to see?

Independent variables: record statistics



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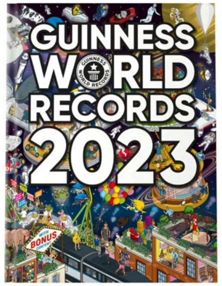
Independent variables: record statistics



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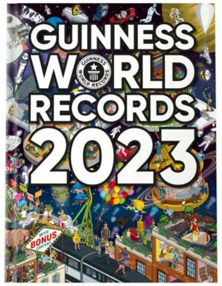
Independent variables: record statistics



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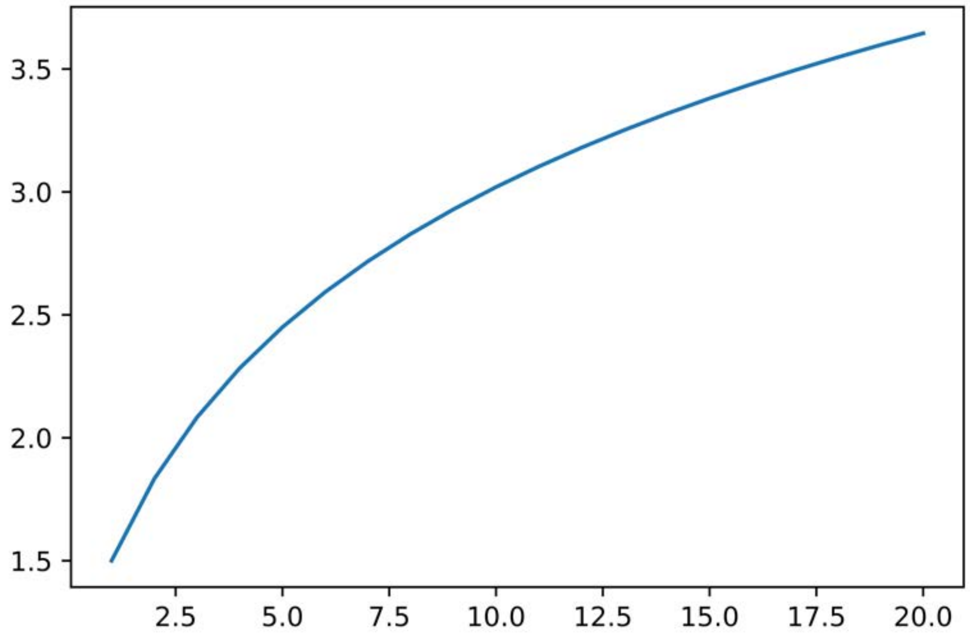
UNIVERSAL!

Independent variables: record statistics

$$\langle N_R \rangle \approx \log(N)$$

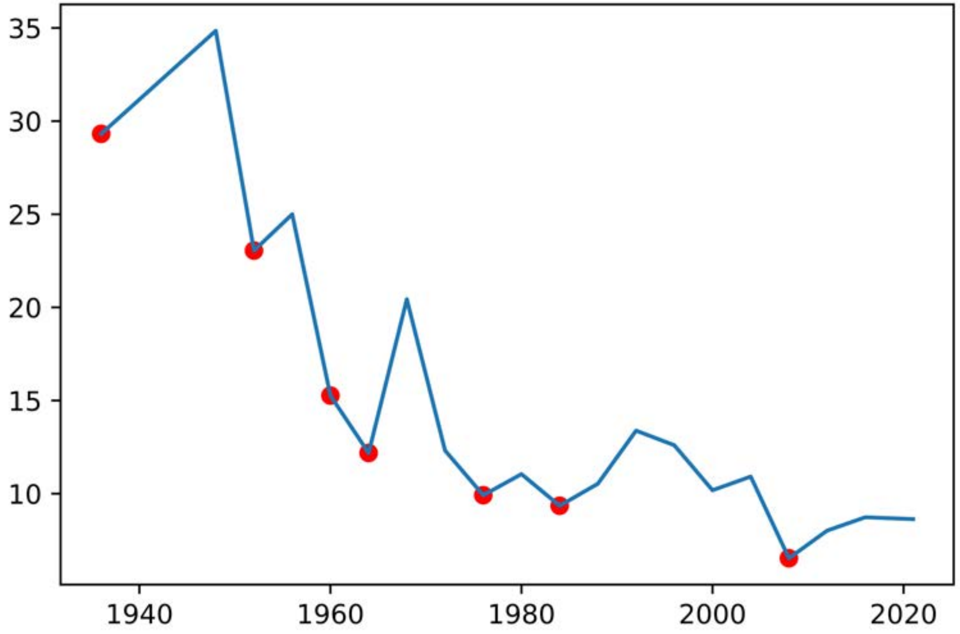
$\langle N_R \rangle$



N

Marathon record statistics

Time above
2 hours [min]

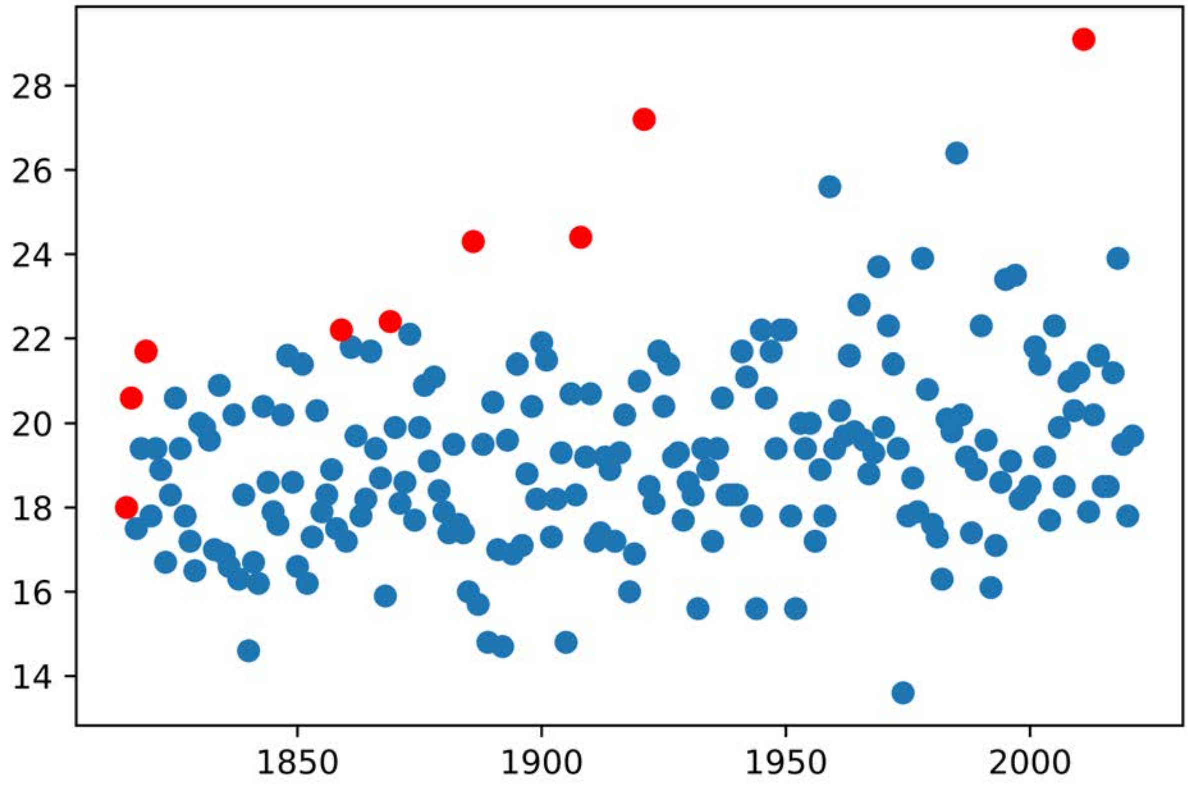


Year

In the last 20 editions of the Olympics → 7 Marathon records

Data from https://olympics.com/en/

Maximal temperature in October



Temperature [°C]

Year

In the last 200 years → 9 records

Data from https://www.geog.ox.ac.uk/

Correlated systems

Correlated random variables

$$P(X_1, \ldots, X_N) \neq \prod_{i=1}^{N} p(X_i)$$

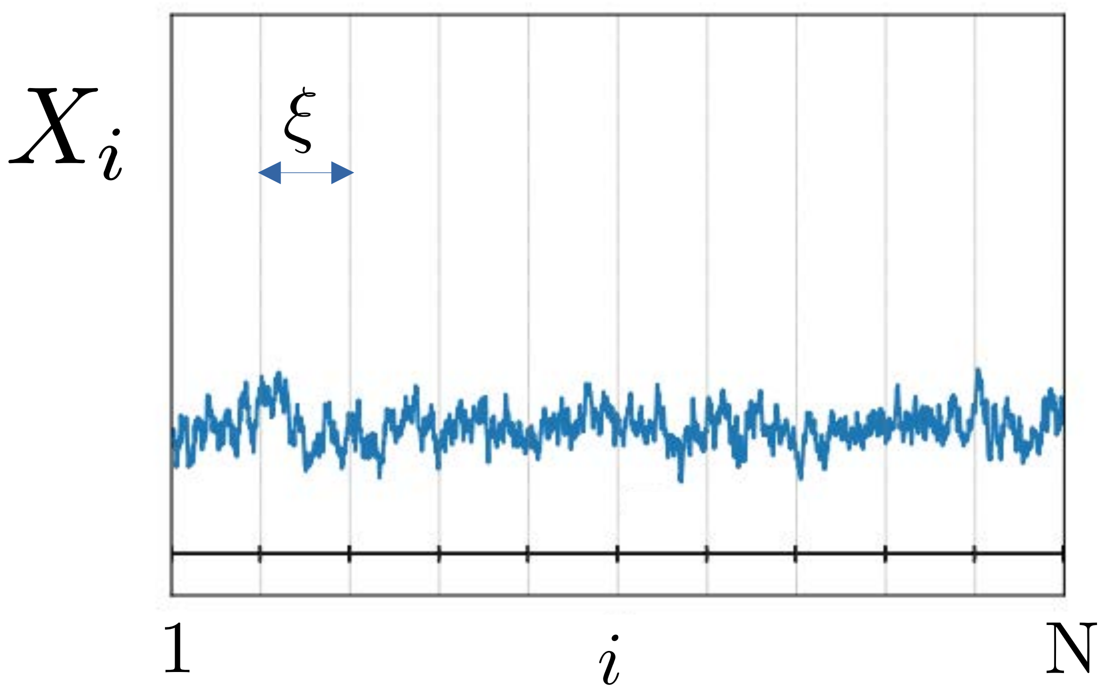
No general technique!

Weakly correlated random variables

$$N \gg \xi$$

correlations

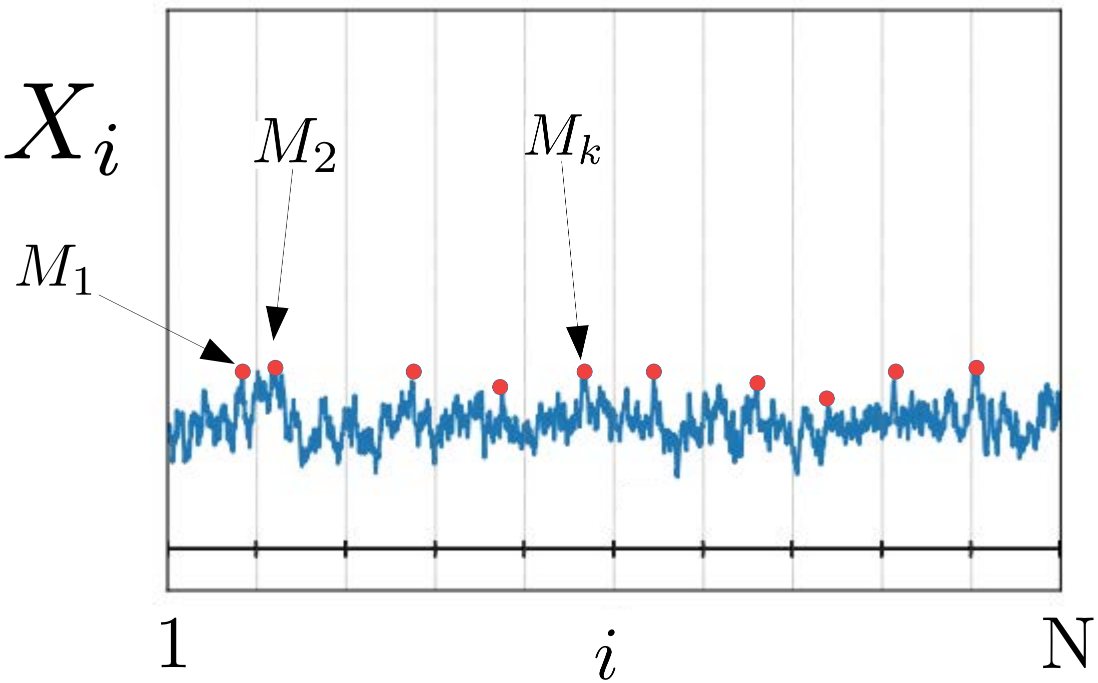
$$\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \sim e^{-|i-j|/\xi}$$



Weakly correlated random variables

i.i.d. problem

$$M = \max_k M_k$$



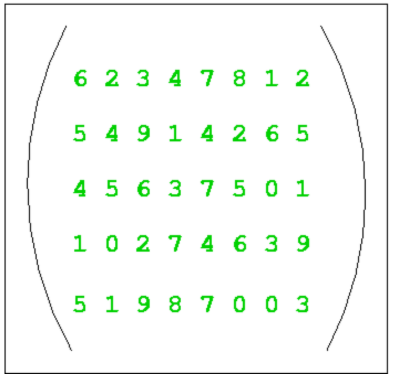
Exactly solvable models

Strongly correlated random variables

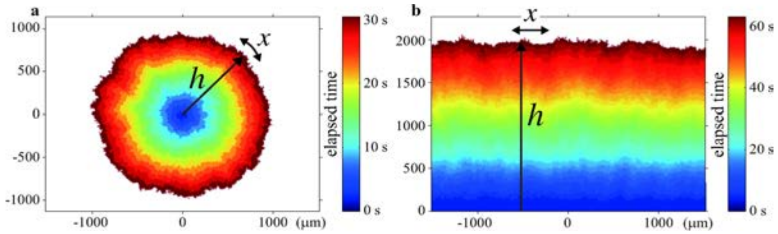
Random Matrices [1,2]

Fluctuating interfaces [3]

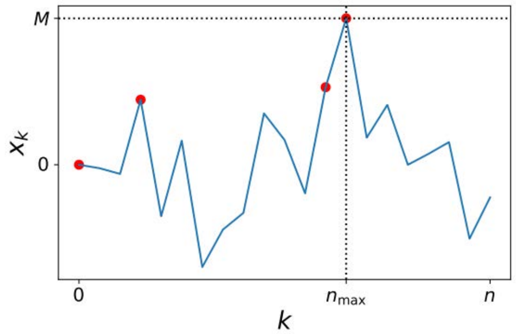
Random walks



From www.sissa.it



From Takeuchi et al., Scientific reports (2011).



[1] C. A. Tracy and H. Widom, Commun. Math. Phys. 159, 151 (1994), 177, 727 (1996).
[2] S. N. Majumdar and G. Schehr, J. Stat. Mech. Theory Exp. P01012 (2014).
[3] S. N. Majumdar and A. Comtet, *Phys. Rev. Lett.* 92, 225501 (2004).

Exactly solvable models

Strongly correlated random variables

Random Matrices [1,2]

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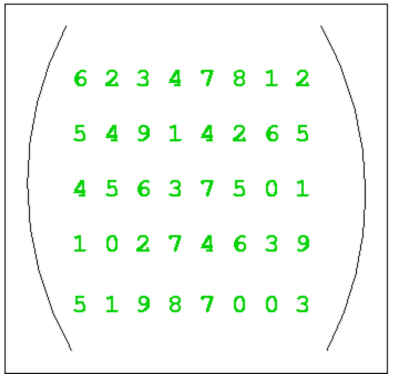
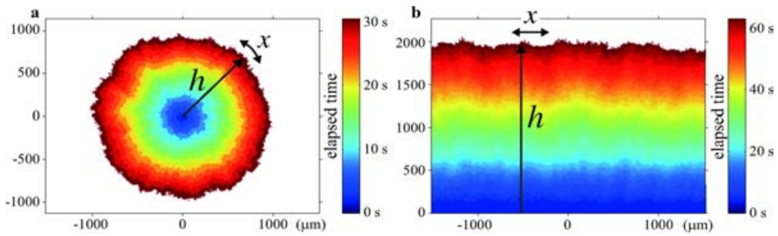
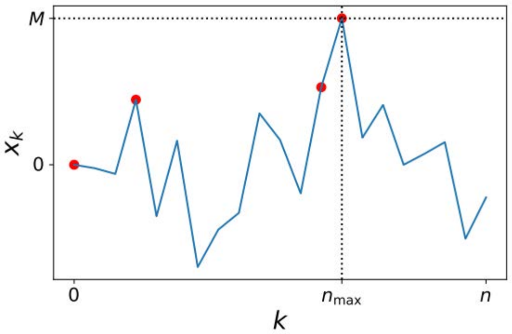


From www.sissa.it

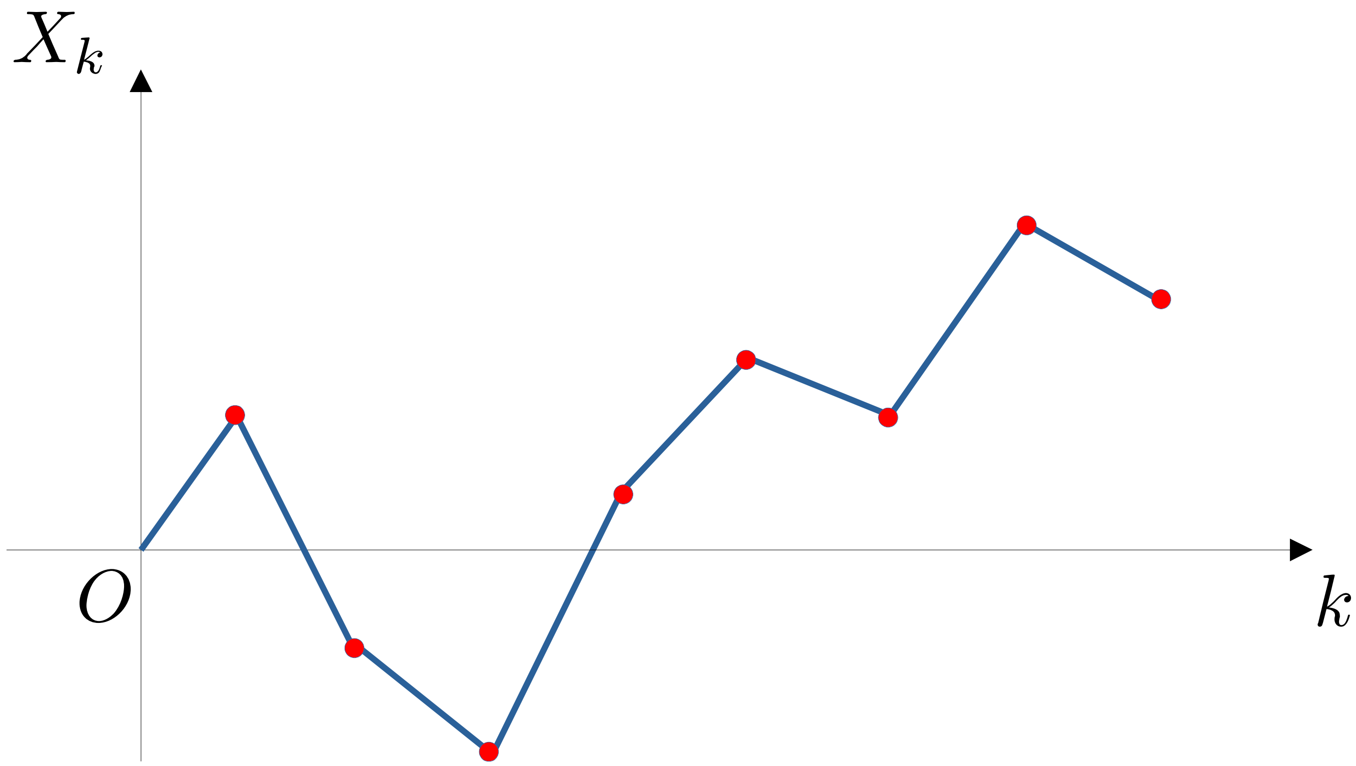


From Takeuchi et al., Scientific reports (2011).



[1] C. A. Tracy and H. Widom, Commun. Math. Phys. 159, 151 (1994), 177, 727 (1996).
[2] S. N. Majumdar and G. Schehr, J. Stat. Mech. Theory Exp. P01012 (2014).
[3] S. N. Majumdar and A. Comtet, *Phys. Rev. Lett.* 92, 225501 (2004).

Random walks

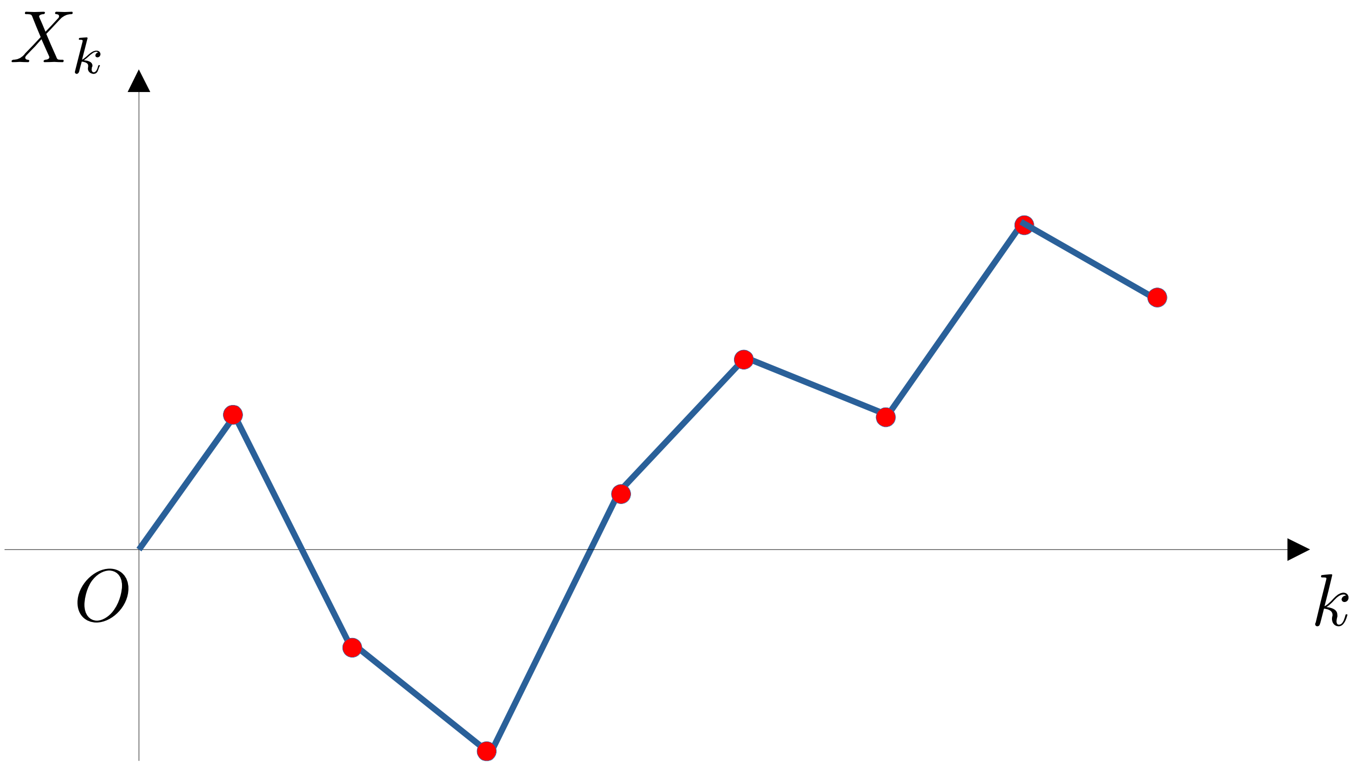


$$X_k = X_{k-1} + \eta_k$$

Independent and identically distributed jumps

$$\eta_k \sim p(\eta)$$

Random walks



$$X_k = X_{k-1} + \eta_k$$

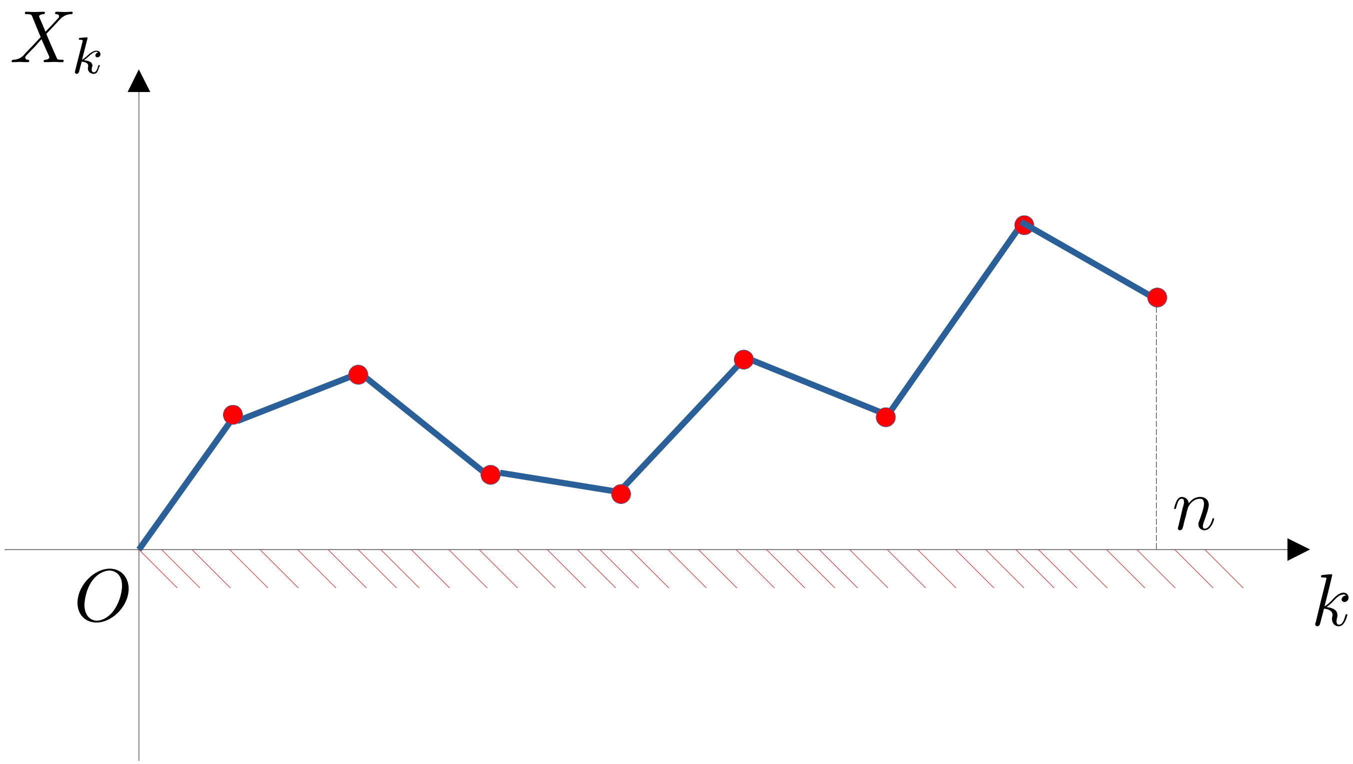
Independent and identically distributed jumps

$$\eta_k \sim p(\eta)$$

$$P(X_1, X_2, \ldots, X_N) = \prod_{i=1}^{N-1} p(X_{i+1} - X_i)$$

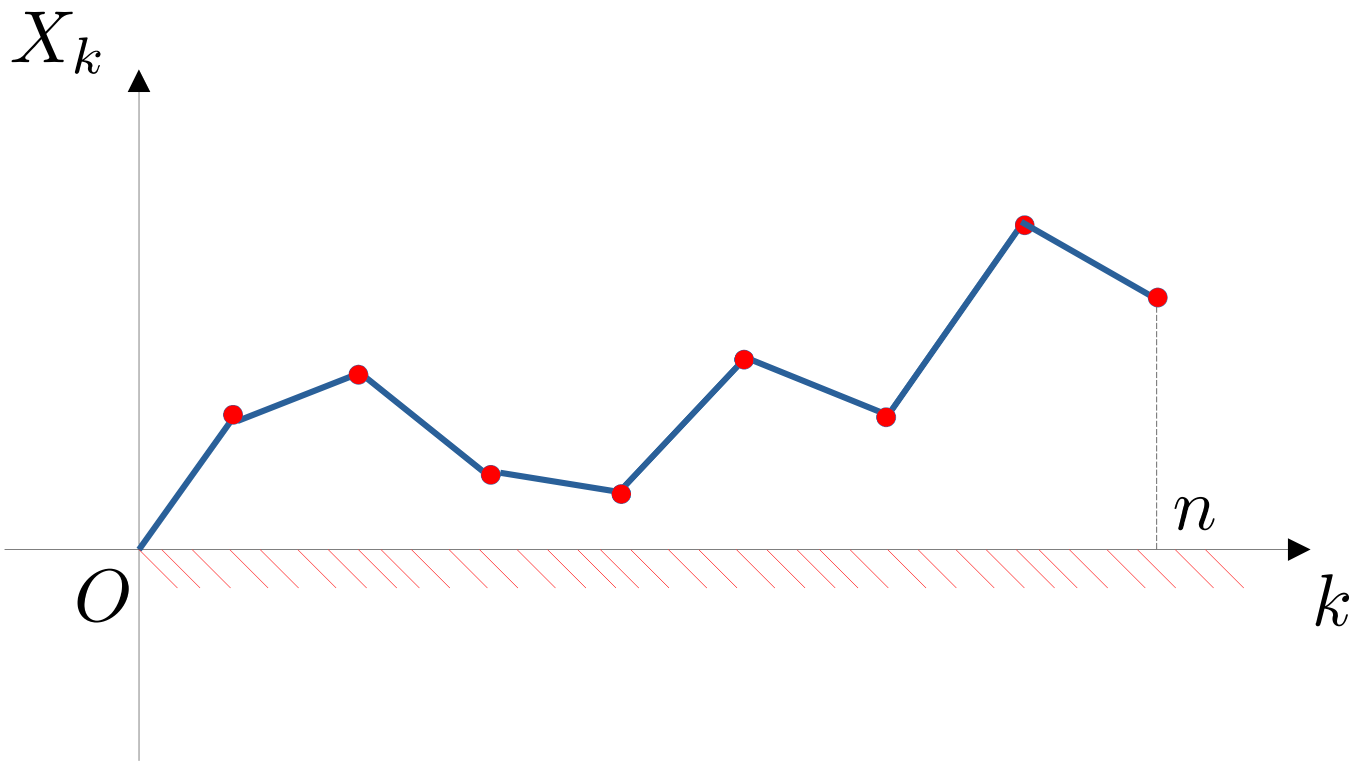
Survival probability

Survival probability



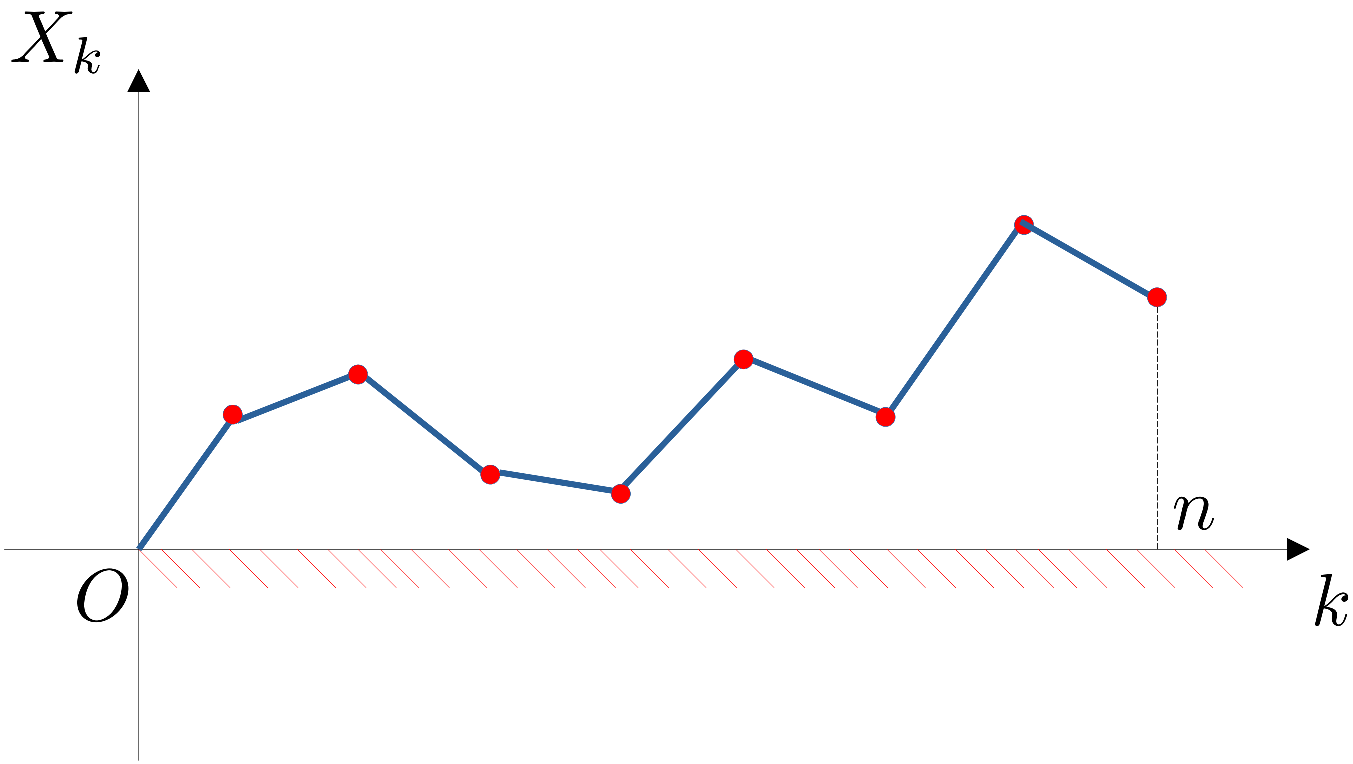
$$q_n = \text{Prob.}(X_1 \geq 0, X_2 \geq 0, \ldots, X_n \geq 0 | X_0 = 0)$$

Survival probability



$$
\begin{aligned}
q_n &= \text{Prob.}(X_1 \geq 0, X_2 \geq 0\,, \ldots\,, X_n \geq 0 | X_0 = 0) \\
&= \int_{-\infty}^{\infty} d\eta_1\,\ldots \int_{-\infty}^{\infty} d\eta_n\,\theta(\eta_1)\theta(\eta_1+\eta_2)\ldots\theta(\eta_1+\eta_2+\ldots+\eta_n)\prod_{i=1}^{n} p(\eta_i)
\end{aligned}
$$

Sparre Andersen theorem



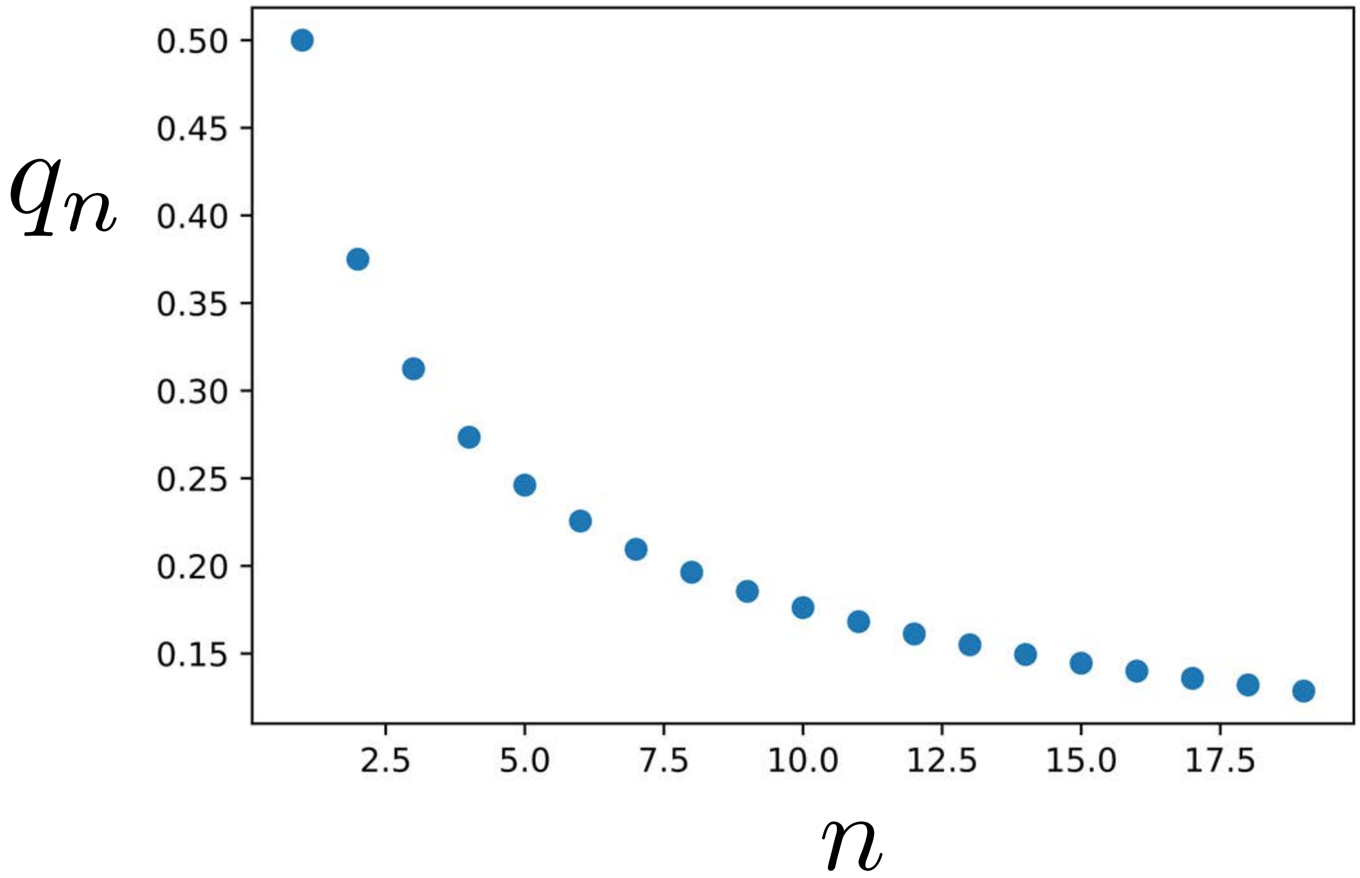
$$q_n = \binom{2n}{n} 2^{-2n}$$

UNIVERSAL!

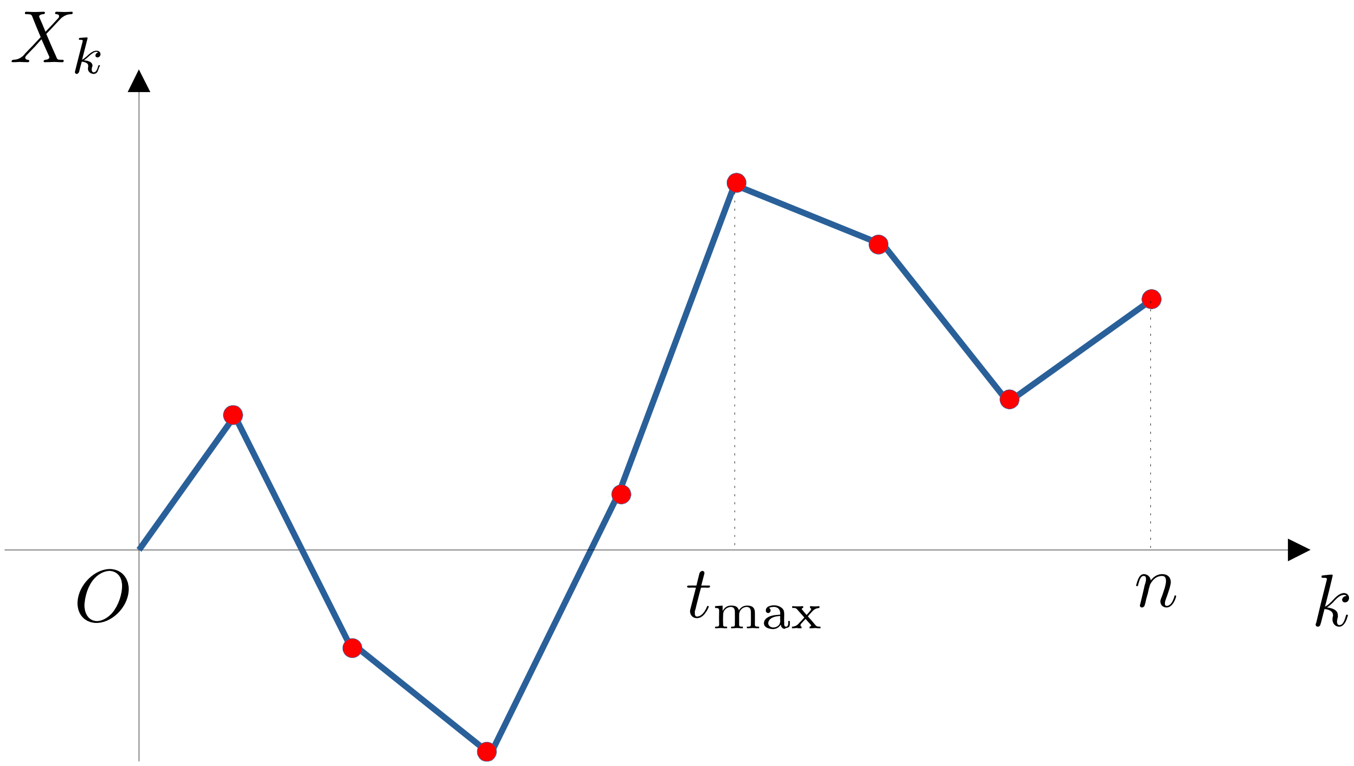
Independent of ***p(η)*** for any ***n***

[1] E. Sparre Andersen, Math. Scand. **2**, 195 (1954).

Sparre Andersen theorem



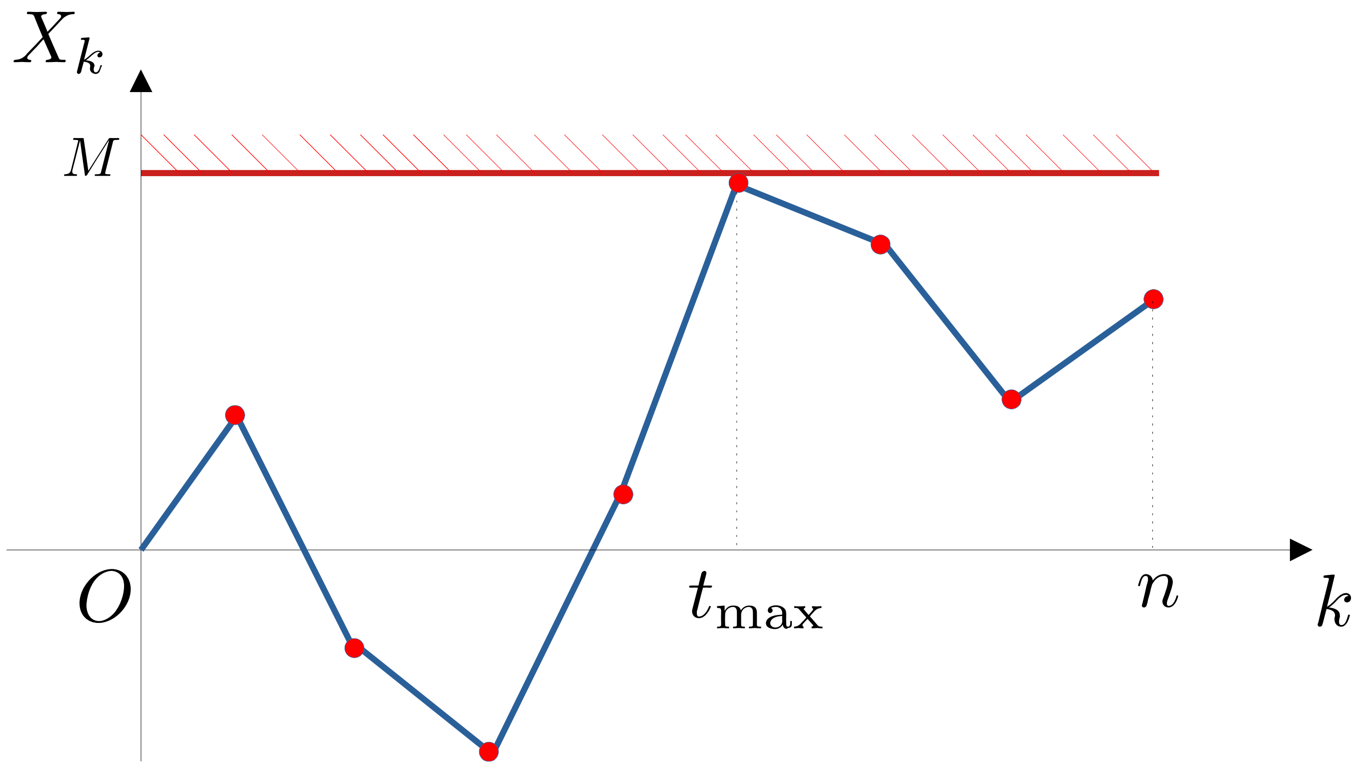
Time of the maximum



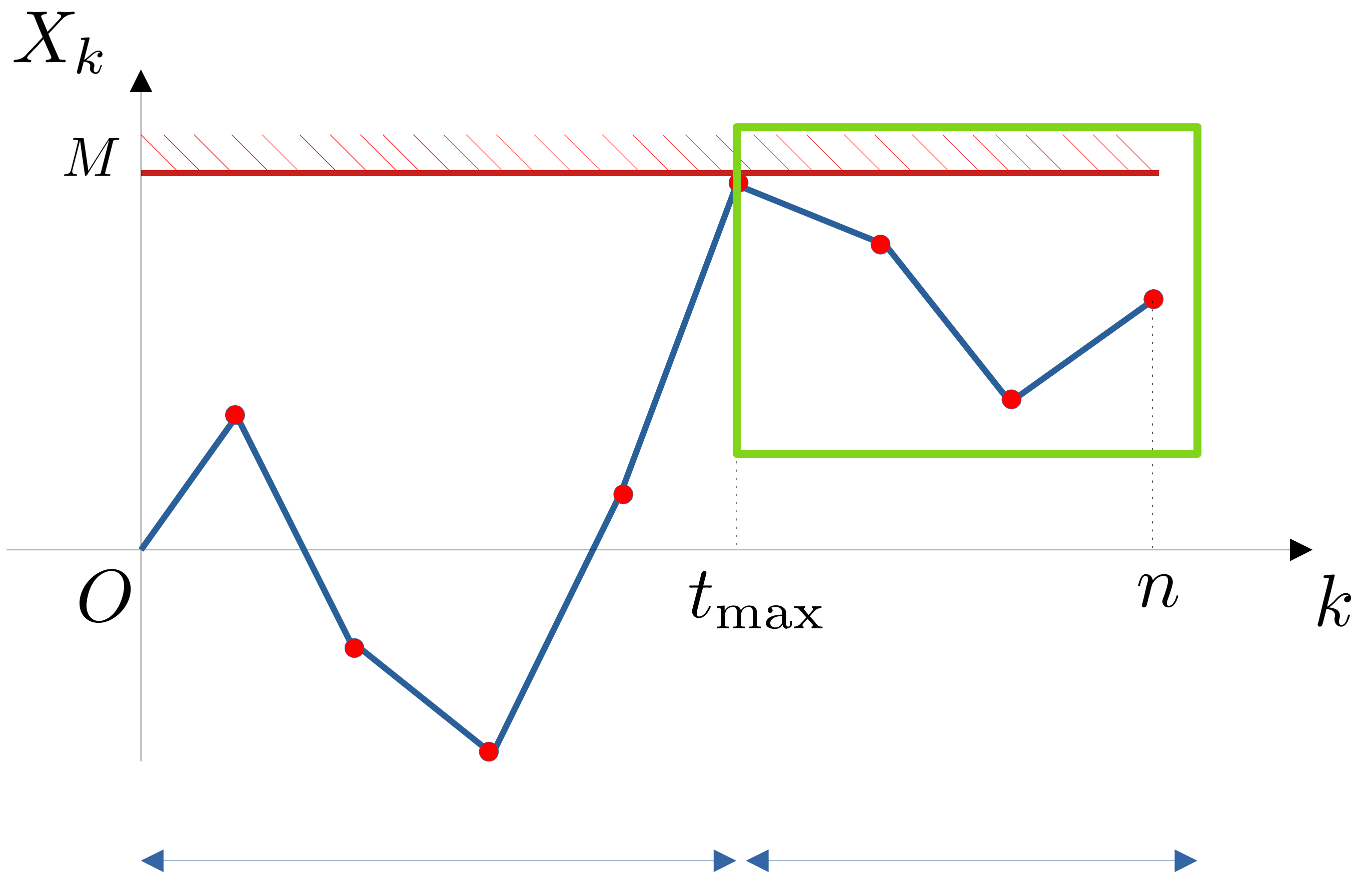
$$P(t_{\max}|n) = ?$$

Applications to finance, sports,...

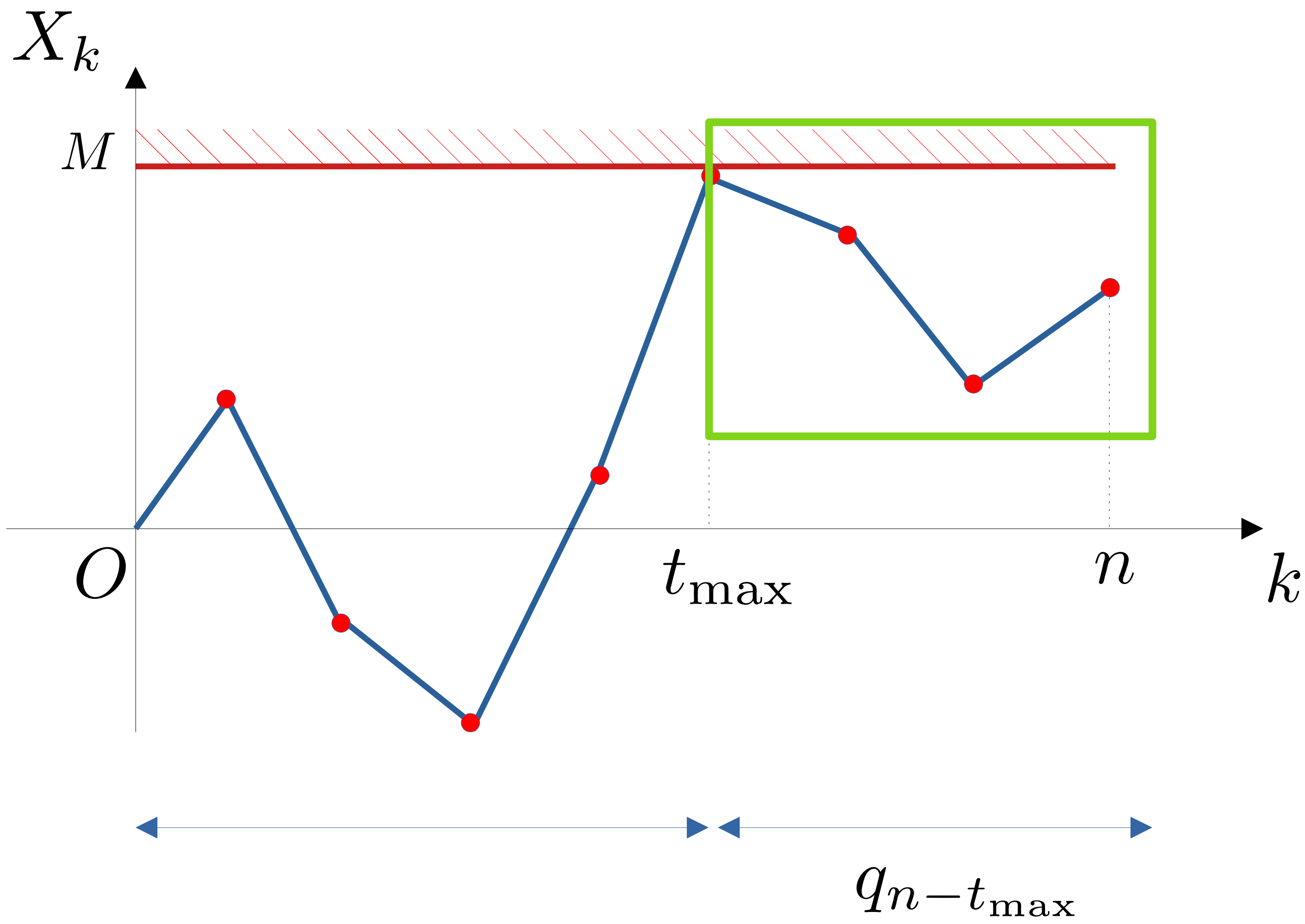
Time of the maximum



Time of the maximum

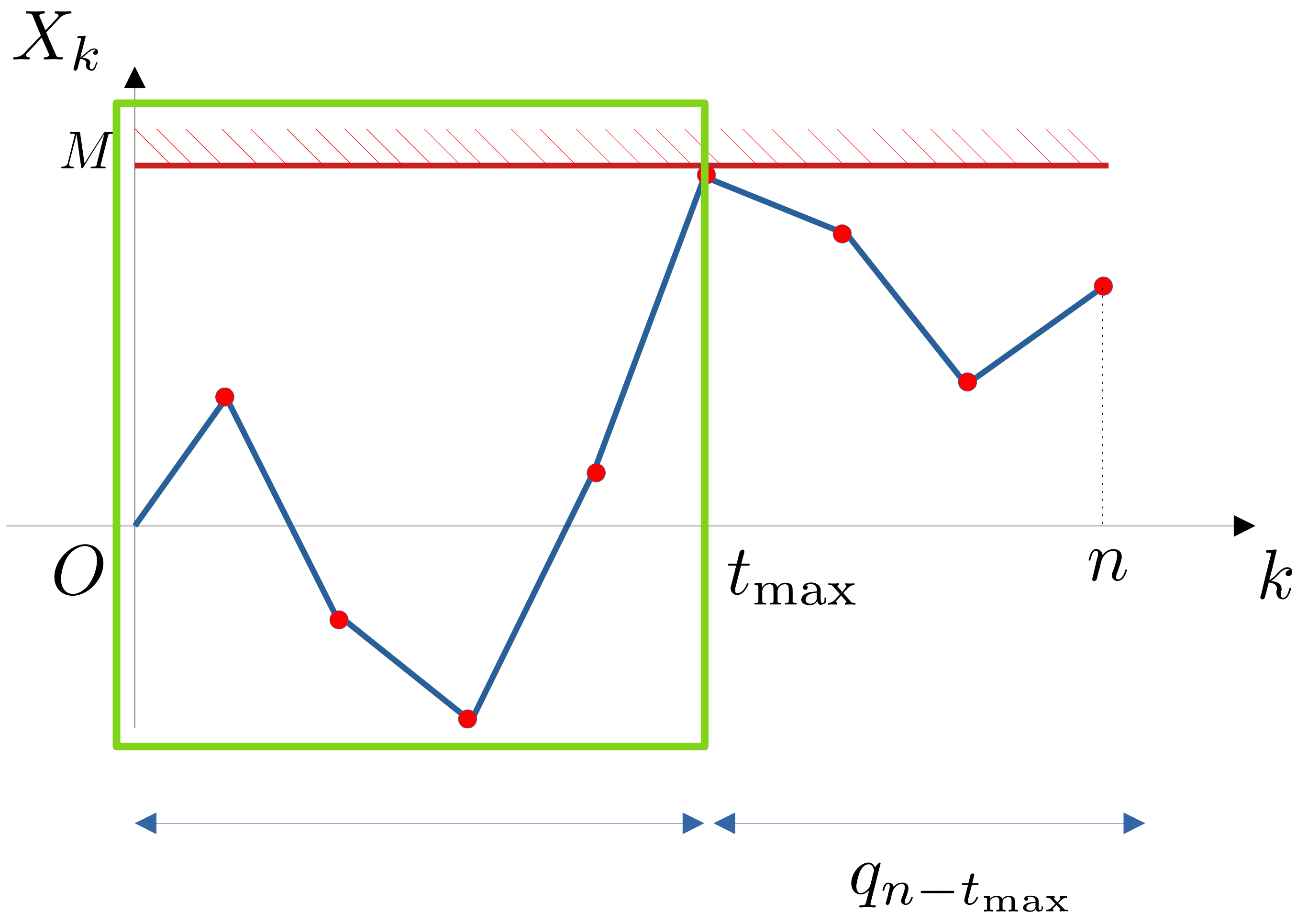


Time of the maximum



$$q_k = \binom{2k}{k} 2^{-2k}$$

Time of the maximum



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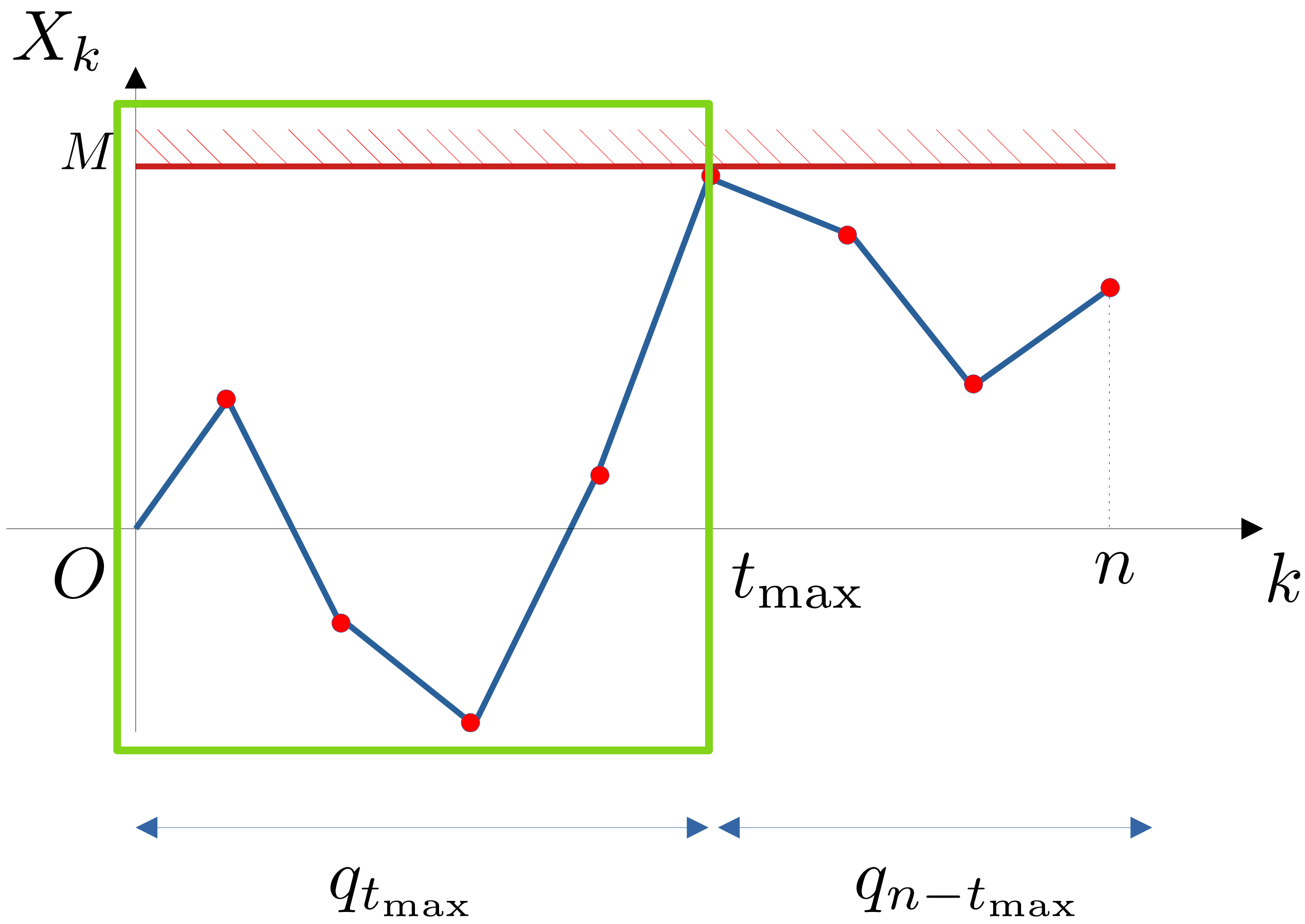
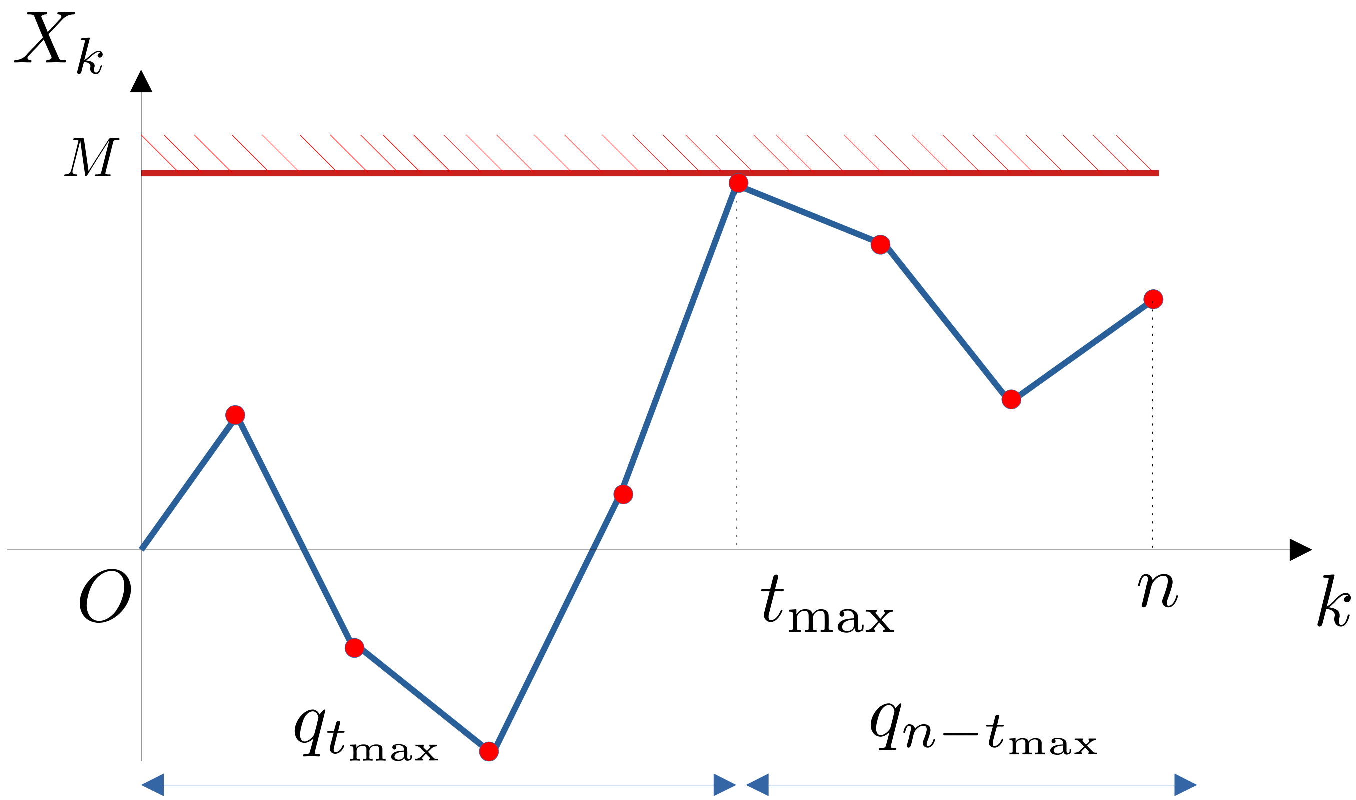
Time of the maximum



$$q_k = \binom{2k}{k} 2^{-2k}$$

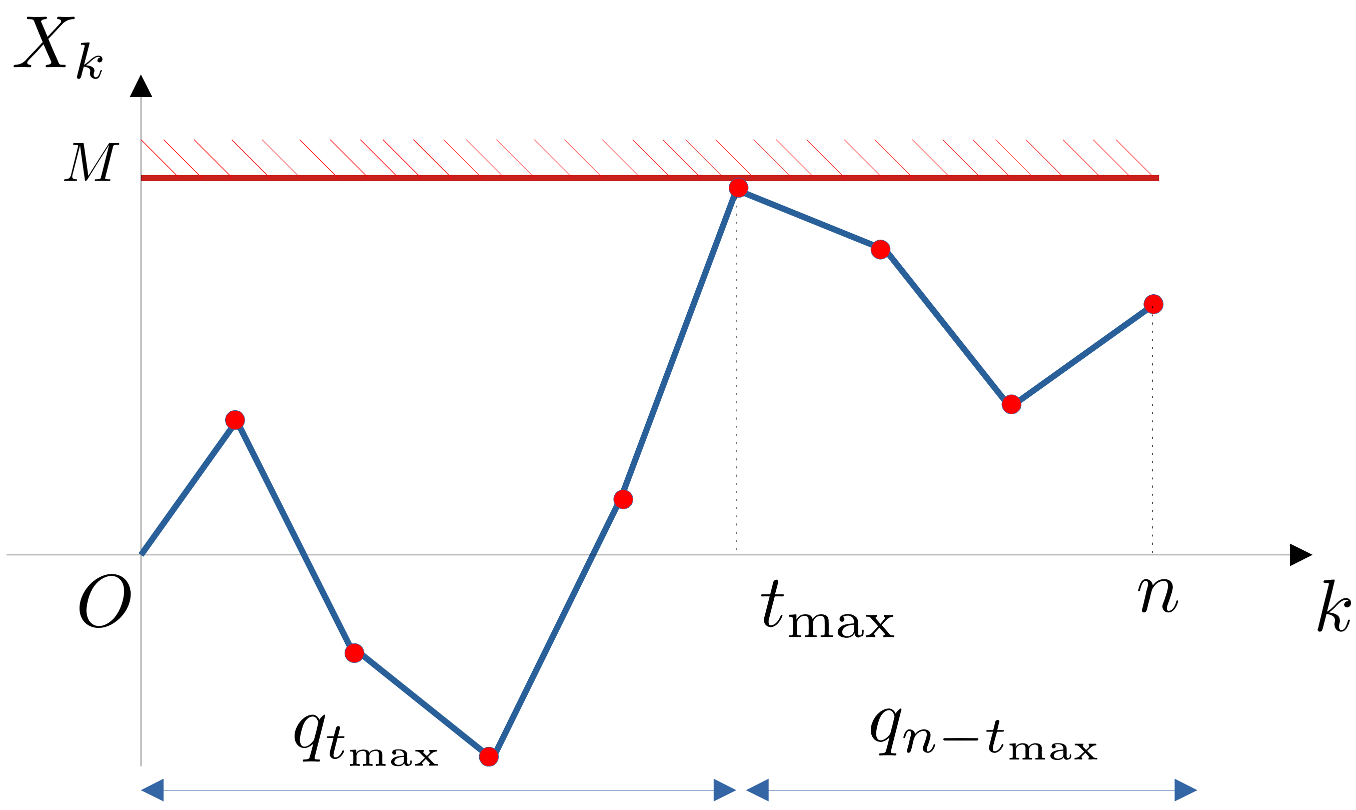
Time of the maximum



$$q_k = \binom{2k}{k} 2^{-2k}$$

$$P(t_{\max}|n) = q_{t_{\max}} q_{n-t_{\max}}$$

Time of the maximum



UNIVERSAL!

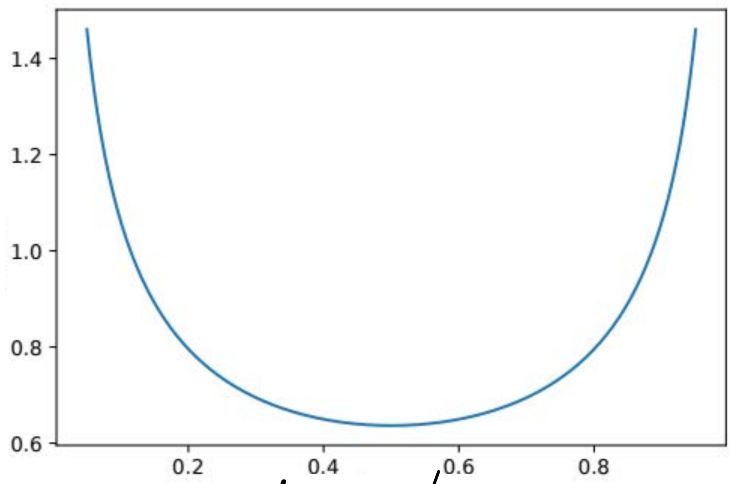
Independent of ***p(η)*** for any ***n***

$$q_k = \binom{2k}{k} 2^{-2k}$$

$$\boxed{P(t_{\max}|n) = q_{t_{\max}} q_{n-t_{\max}}}$$

Time of the maximum of a random walk

$P(t_{\max}|n)$

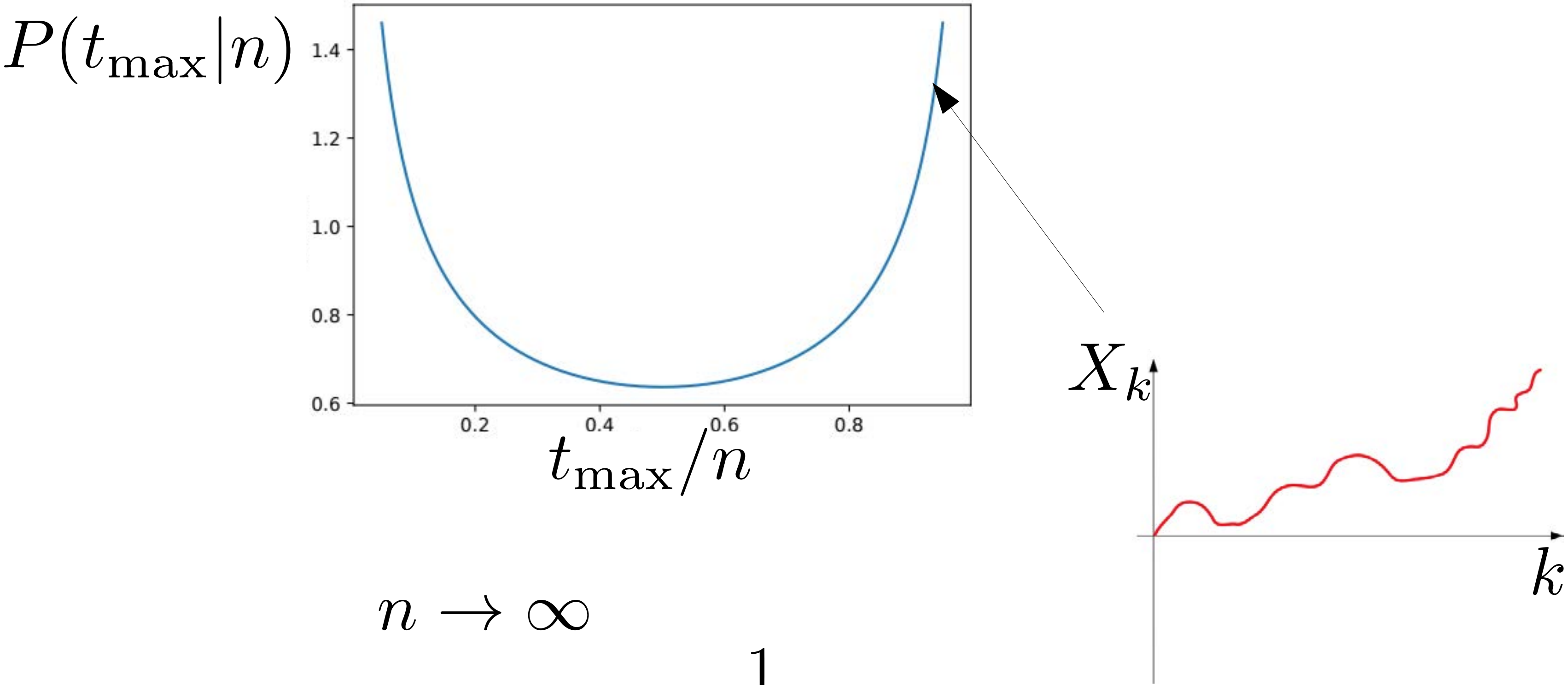


$t_{\max}/n$

$$n \to \infty$$

$$P(t_{\max}|n) \approx \frac{1}{\pi\sqrt{t_{\max}(n - t_{\max})}}$$

Time of the maximum of a random walk



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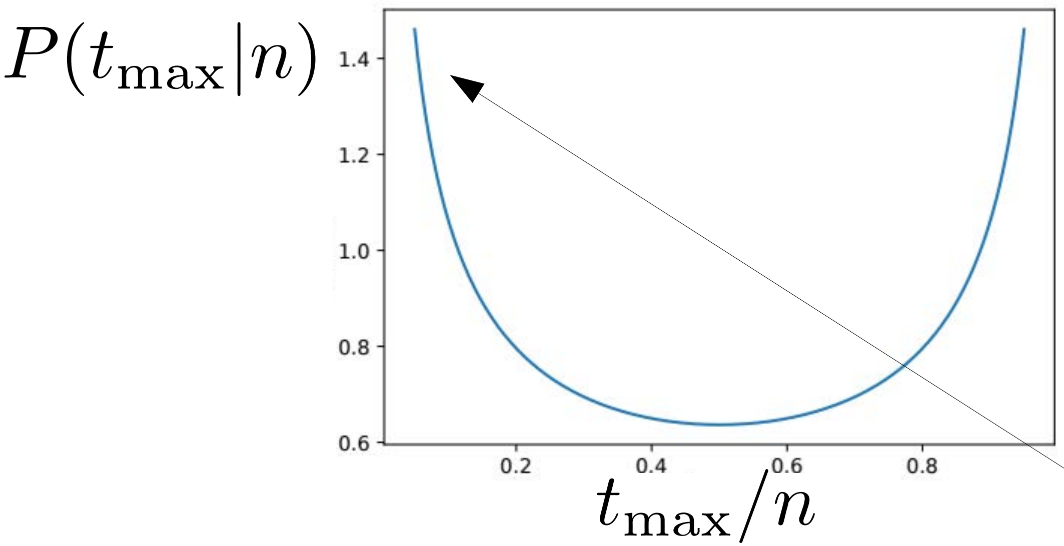
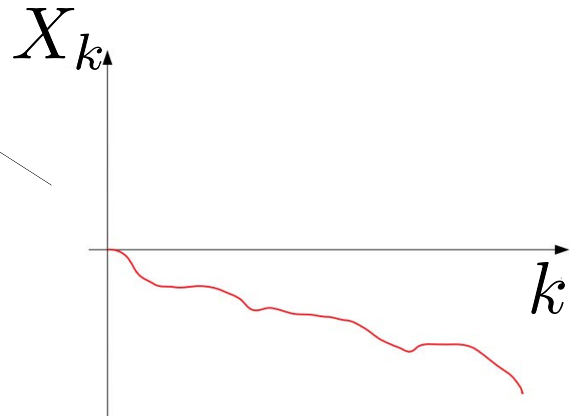
Time of the maximum of a random walk





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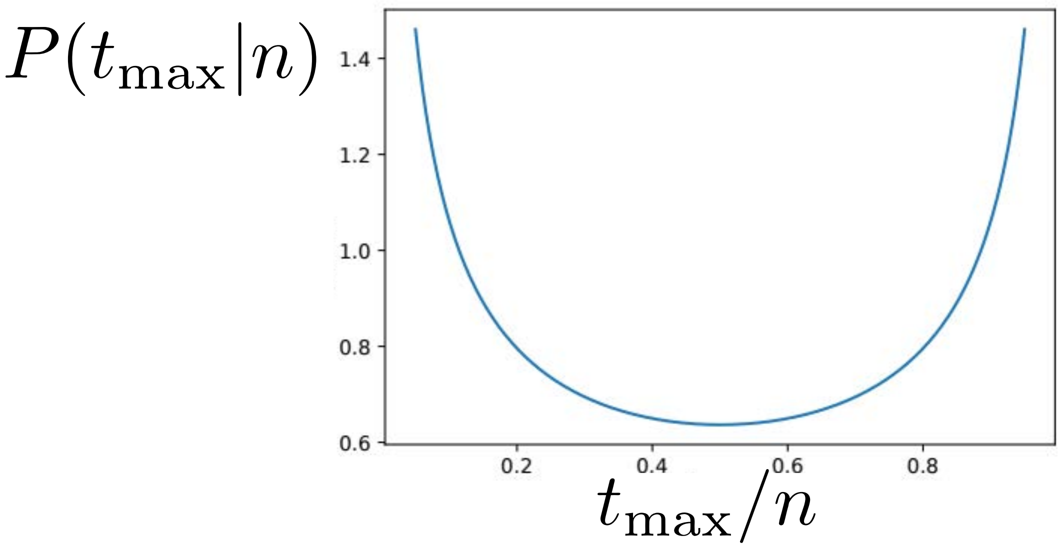
Time of the maximum of a random walk

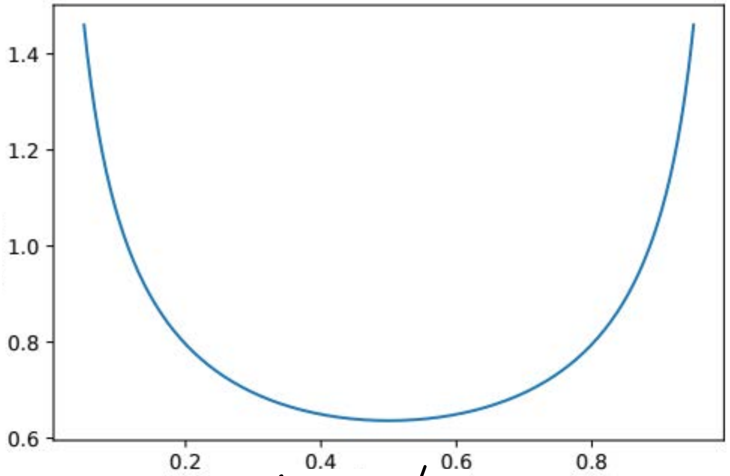


$$n \to \infty$$

$$P(t_{\max}|n) \approx \frac{1}{\pi\sqrt{t_{\max}(n - t_{\max})}}$$

Time of the minimum of a random walk

$P(t_{\min}|n)$

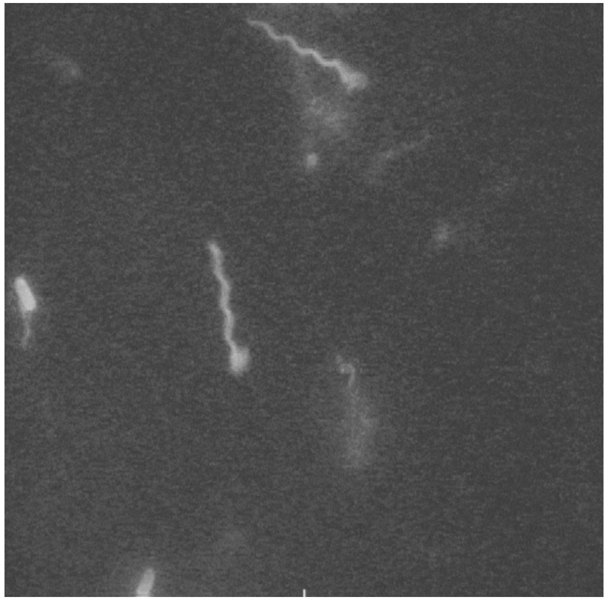


$t_{\min}/n$

$$P(t_{\min}|n) \approx \frac{1}{\pi\sqrt{t_{\min}(n - t_{\min})}}$$

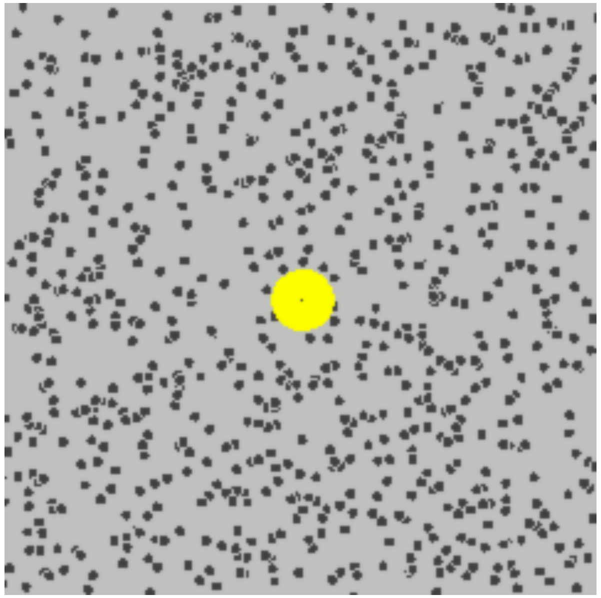
Active Particles

Active vs passive matter



Active Motion of
E. Coli bacteria
(Berg et al.)

- Persistent motion (absorbing energy from the environment)
- Out of equilibrium
- Alive

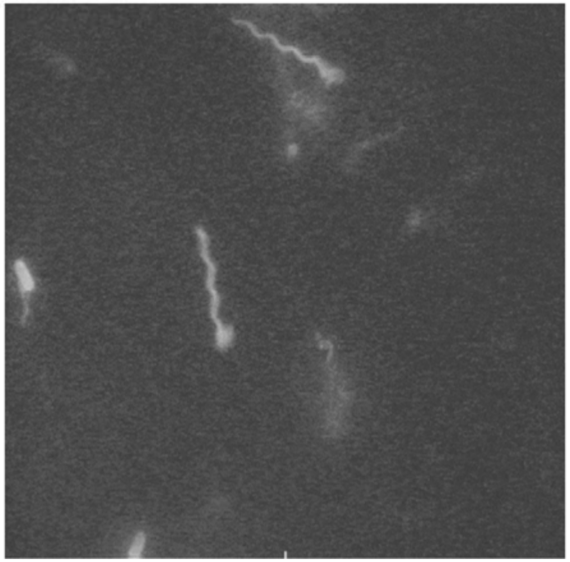


(Passive) Brownian
Motion

- Random motion (collisions with smaller molecules)
- Equilibrium
- Dead

Run-and-tumble particle (RTP) model

- Motion in d dimensions
- Random velocity $v \sim W(v)$
- Tumbling rate γ



E. Coli bacteria (Berg et al.)

[1] E. Orsingher, Random Oper. and Stoch. Equ. 3, 9 (1995).

Run-and-tumble particle (RTP) model

RTP model

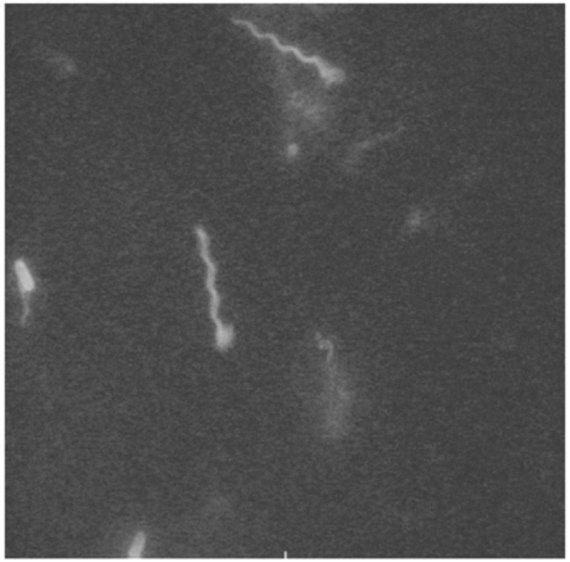
- Motion in **d** dimensions
- Random velocity **v~ W(v)**
- Tumbling rate γ

What is the **survival probability S(t)**?

S(t) = probability that the x-component of the particle does not change sign up to time t

Known for **d=1** and **constant velocity [1]**

$$S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0(\gamma t/2) + I_1(\gamma t/2) \right)$$

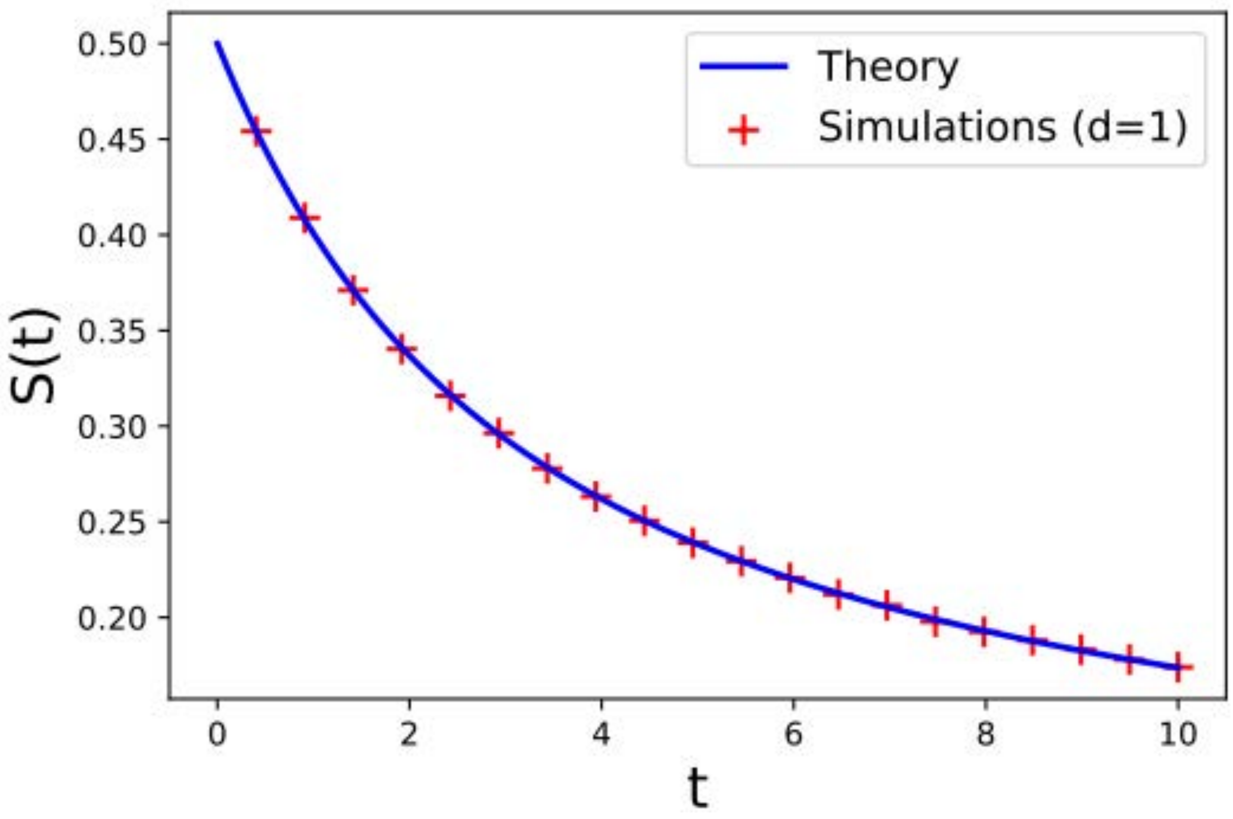


E. Coli bacteria (Berg et al.)

[1] E. Orsingher, Random Oper. and Stoch. Equ. 3, 9 (1995).

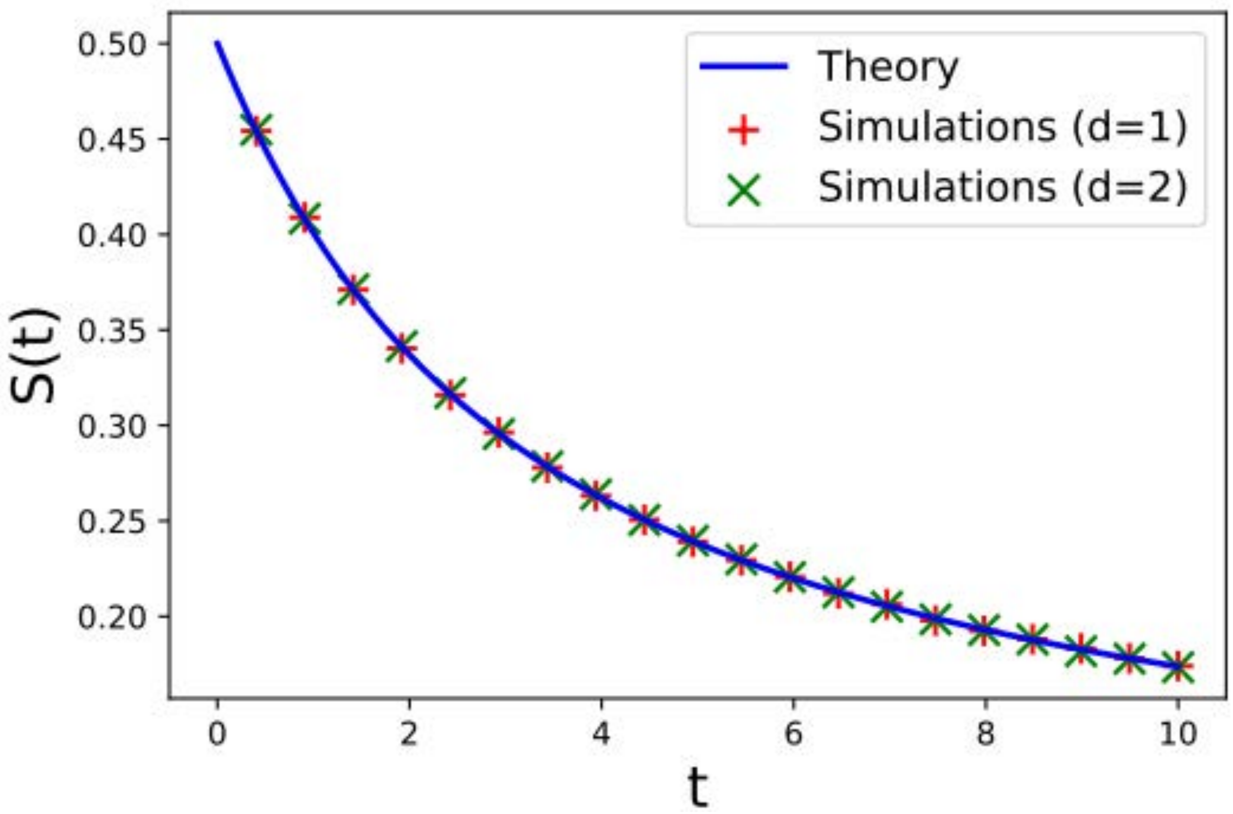
Numerical simulations

$$S(t) = \frac{1}{2}e^{-\gamma t/2}\left(I_0(\gamma t/2) + I_1(\gamma t/2)\right)$$



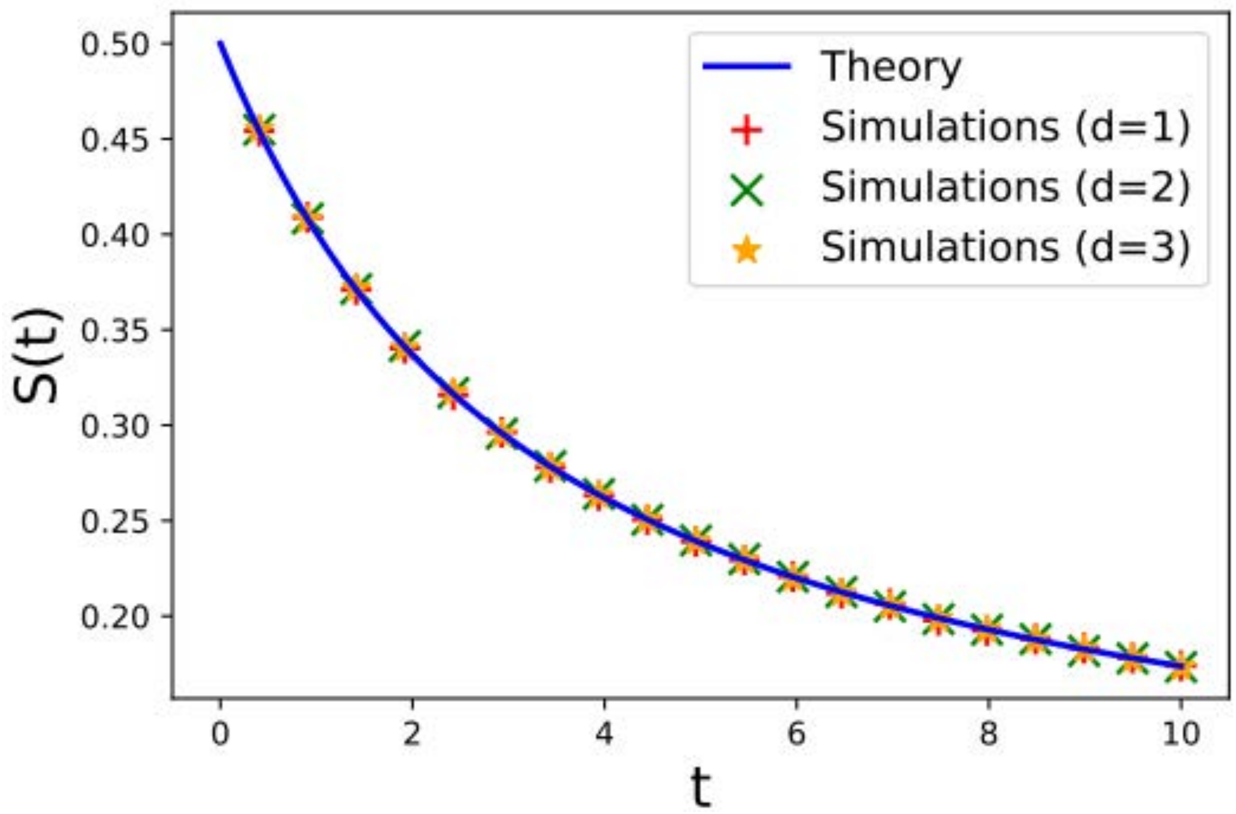
Numerical simulations

$$S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0(\gamma t/2) + I_1(\gamma t/2) \right)$$



Numerical simulations

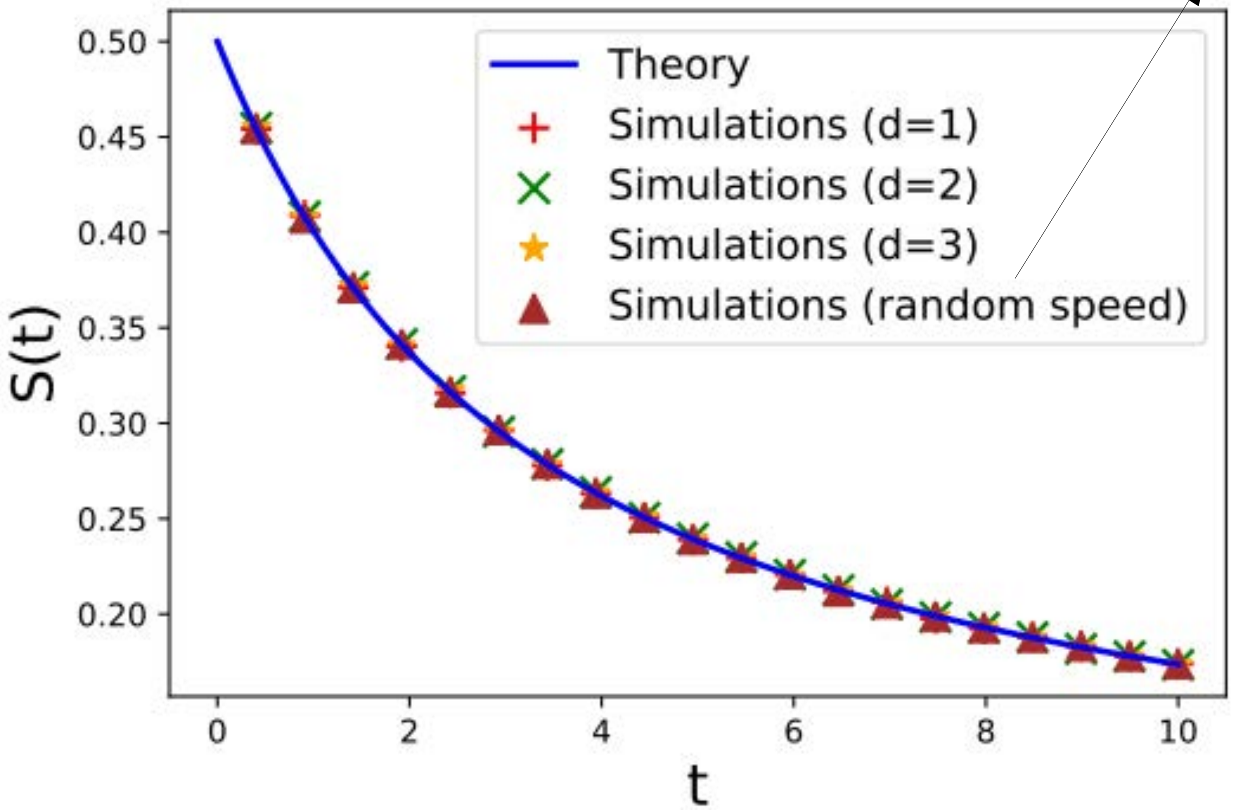
$$S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0(\gamma t/2) + I_1(\gamma t/2) \right)$$



Numerical simulations

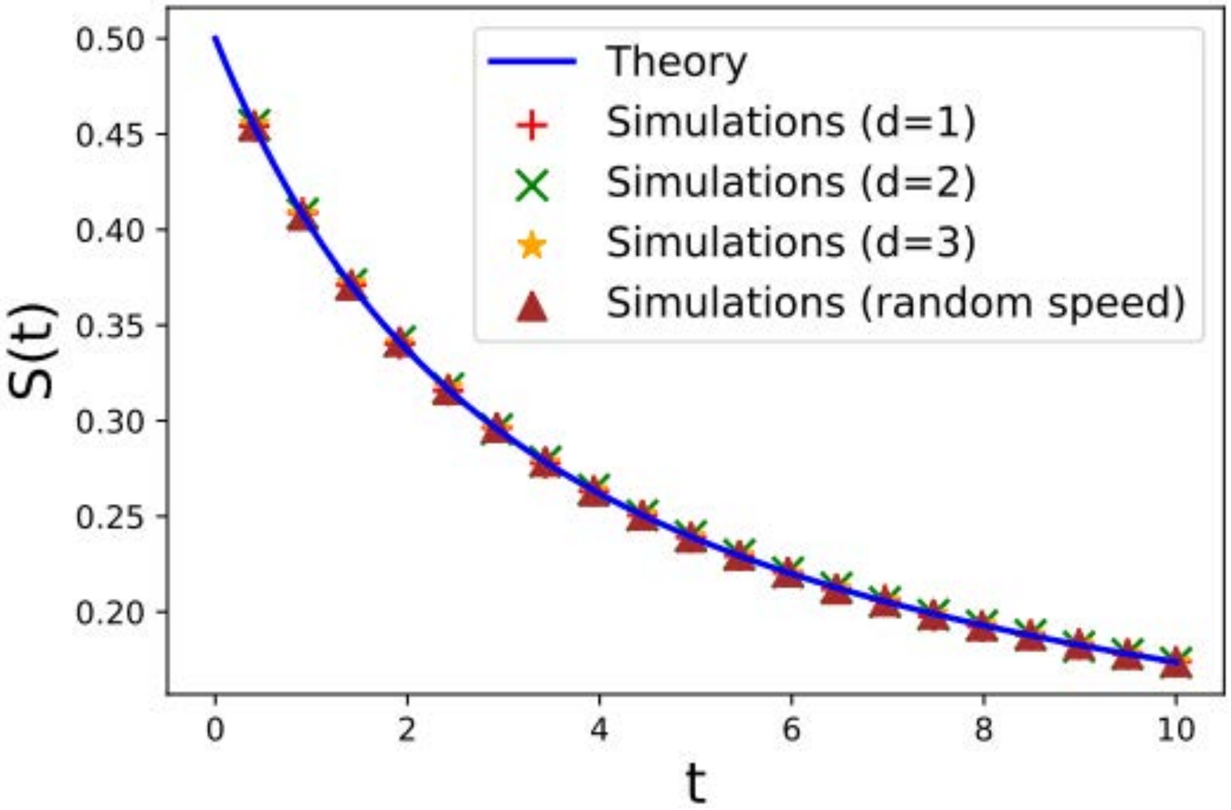
$$S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0(\gamma t/2) + I_1(\gamma t/2) \right)$$

$$W(v) = \frac{2}{\pi(1+v^2)}$$



Numerical simulations

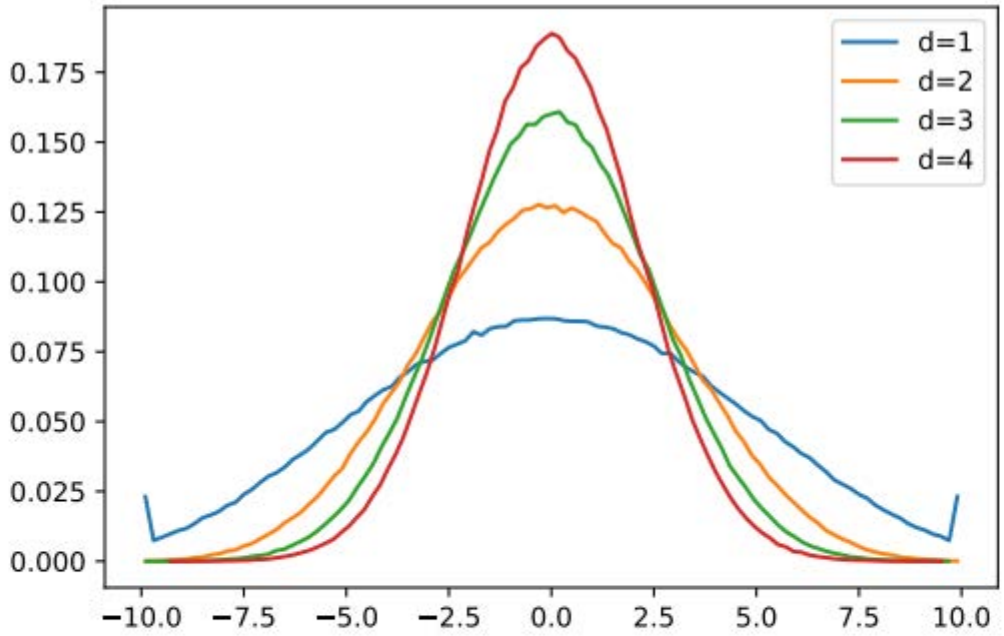
Universality



Position distribution

Not universal!

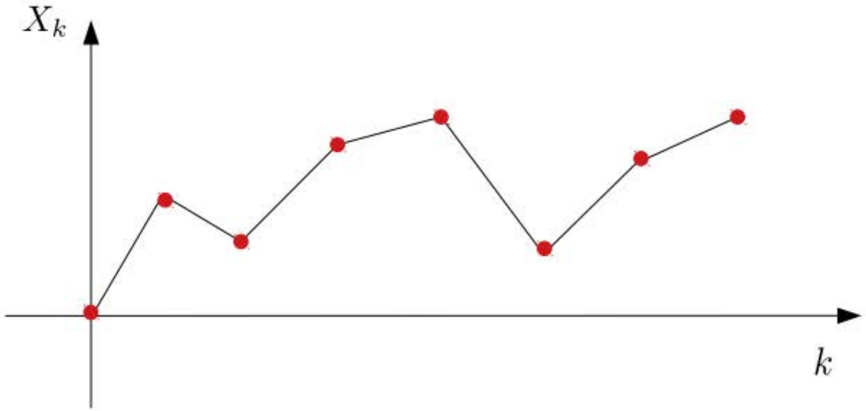
$P(X,t)$



X

$t = 10, \gamma = 1$

Sparre Andersen theorem



$$q_n = \binom{2n}{n} 2^{-2n}$$

Independent of ***p(η)*** for any ***n***

It is not obvious how to apply this result to the RTP model

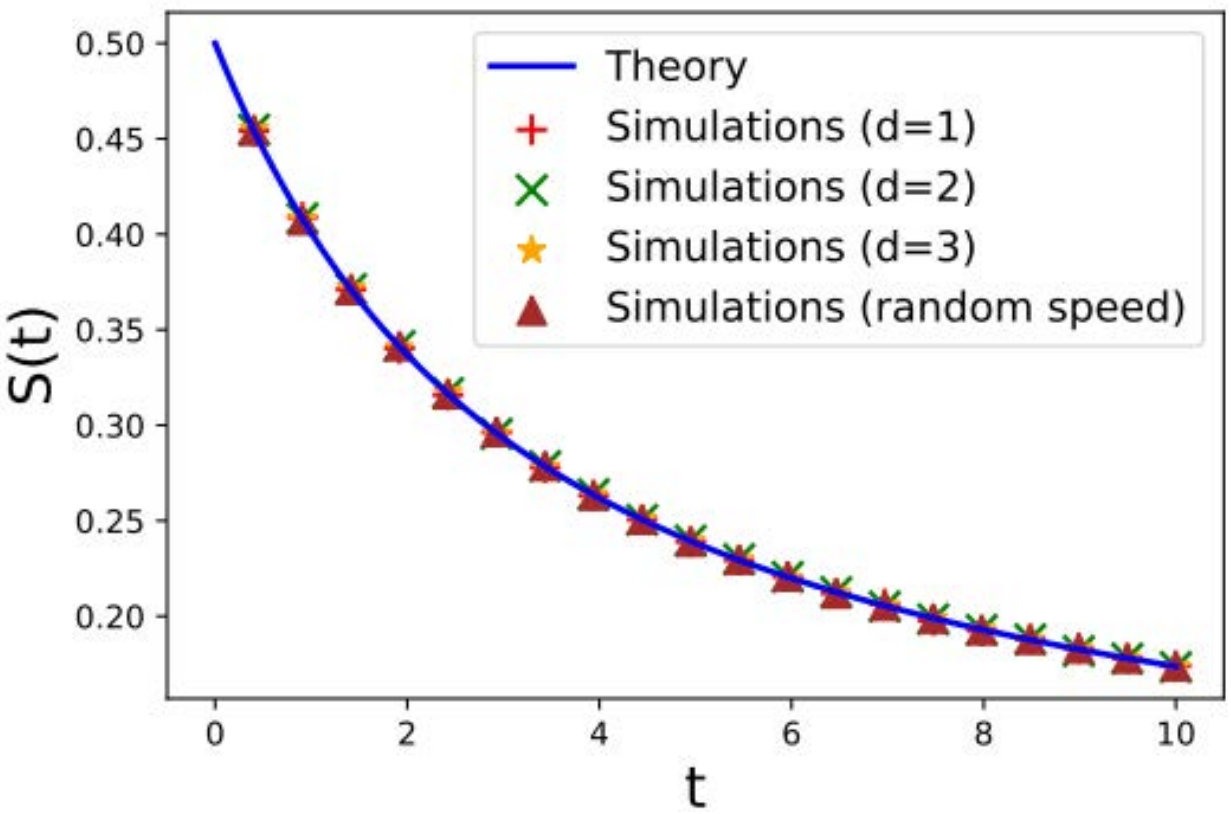
↓

Mapping to a discrete-time random walk (in Laplace space)

[1] E. Sparre Andersen, Math. Scand. **2**, 195 (1954).

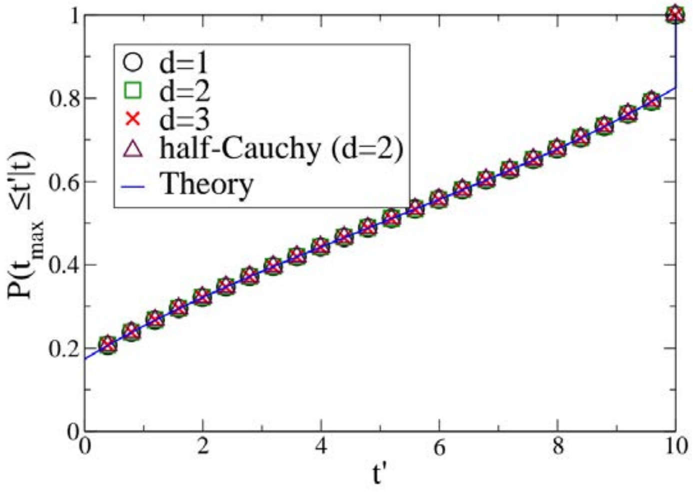
Sparre Andersen Theorem

Universality

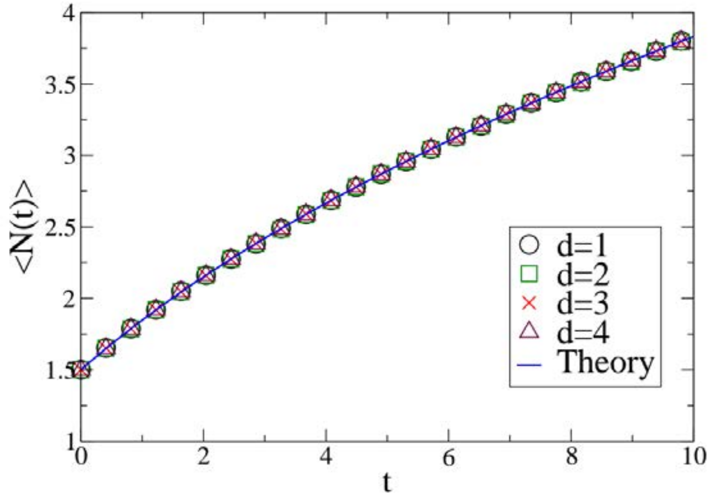


Extreme Value Statistics

Time of the maximum [1]



Average number of records [1]



Also universal!

[1] FM, P. Le Doussal, S. N. Majumdar, and G. Schehr, Phys. Rev. E 102, 042133 (2020).

Extreme Value Statistics

- Using simple models, we can learn a lot about the statistics of extreme events, records, ...

- Many applications to physics, finance, evolution theory, ...

- Often we find universal results, independent of the details of the model

- Can we find a general theory of rare events in correlated systems?

Thank you for the attention